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4.3) Morphisms:

$\Lambda_1, \Lambda_2 \subset \mathbb{C}_\infty$ lattices of rank r
and $c \in \mathbb{C}_\infty$, $c\Lambda_1 \subset \Lambda_2$ morphism of lattices

$$\mathrm{Hom}(\Lambda_1, \Lambda_2) := \{ c \in \mathbb{C}_\infty \mid c\Lambda_1 \subset \Lambda_2 \}$$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_1 \arrow[r] \arrow[d, "c"] & \mathbb{C}_\infty \arrow[r, "e_{\Lambda_1}"] \arrow[d, "c"] \arrow[dr, phantom, "\bigcirc"] & \mathbb{C}_\infty \arrow[r] \arrow[d, "\varphi_c"] & 0 \\
0 \arrow[r] & \Lambda_2 \arrow[r] & \mathbb{C}_\infty \arrow[r, "e_{\Lambda_2}"'] & \mathbb{C}_\infty \arrow[r] & 0
\end{tikzcd}$$

$\varphi_c(e_{\Lambda_1}(z)) = e_{\Lambda_2}(cz)$

With $\varphi_c(z) = \underbrace{c \cdot e_{e_{\Lambda_1}(c^{-1}\Lambda_2)}(z)}_{\mathbb{F}_q\text{-linear polynomial of } \deg_\tau = \dim_{\mathbb{F}_q} \frac{c^{-1}\Lambda_2}{\Lambda_1}}$ will do

φ_c is compatible with the φ and φ' structure:

$$\varphi_c \circ \varphi_a^{\Lambda_1} = \varphi_a^{\Lambda_2} \circ \varphi_c$$

Algebraically morphisms can be defined as

$$\mathrm{Hom}(\varphi, \varphi') := \{ f \in \mathbb{C}_\infty[\tau] \mid f\varphi_a = \varphi'_a f \ \forall a \in A \}$$

So we get a category of Drinfeld modules & morphisms

<u>Theorem (Uniformization)</u>: There is a equivalence of categories

$$\left\{\begin{matrix}\text{Drinfeld } A\text{-modules of} \\ \text{rank } r \text{ over } \mathbb{C}_\infty\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}A\text{-lattices in } \mathbb{C}_\infty \text{ of} \\ \text{rank } r\end{matrix}\right\}$$

<u>Proof sketch</u>: Given a Drinfeld module φ, pick $t \in A$ transcendental over $\mathbb{F}_q$, then we must find an $\mathbb{F}_q$-linear power series $e_\Lambda(x) \in \mathbb{C}_\infty[[\tau]]$ s.t.

$$e_\Lambda(tz) = \varphi_t(e_\Lambda(z)) \quad , \; de_\Lambda = 1$$

Solve for the coefficients of $e_\Lambda(z)$, then let $\Lambda := \ker e_\Lambda$. Now show that φ is the Drinfeld module corresponding to Λ. $\square$

4.6) <u>Endomorphisms</u>

$$\mathrm{End}(\varphi) = \mathrm{Hom}(\varphi, \varphi) = \{f \in \mathbb{C}_\infty[\tau] \mid f\varphi_a = \varphi_a f \;\; \forall a \in A\}$$

$$= \mathrm{Cent}_{\mathbb{C}_\infty[\tau]}\, \varphi(A)$$

$$\cong \{c \in \mathbb{C}_\infty \mid c\Lambda \subset \Lambda\}$$

<u>Theorem</u>: $R = \mathrm{End}(\varphi)$ is a domain, projective A-module of rank $s \mid r$. Let $K = \mathrm{Quot}(R) = R \otimes_A F$, then K/F is an imaginary extension, ie only one prime of K lies above ∞. R is an order in K.
$\exists\, 0 \neq f \in \mathrm{Hom}(\varphi, \tilde{\varphi})$ with $\tilde{R} = \mathrm{End}(\tilde{\varphi})$ the integral closure of A in K. In particular, $\tilde{\varphi}$ is a Drinfeld $\tilde{R}$-module

of rank $\frac{r}{s}$.

Proof sketch:

Straightforward, K/F is imaginary because Λ and thus also $\mathrm{End}(\Lambda)$ is discrete in $\mathbb{C}_\infty$. $\square$

If $[K:F]=r$, i.e. $\mathrm{End}(\varphi)$ has rank r over A, we say that φ has complex multiplication by $\mathrm{End}(\varphi)$.

In this case, φ is a rank 1 $\mathrm{End}(\varphi)$-module.

$\Rightarrow$ Get a nice theory of complex multiplication.

4.5) Isomorphisms: $f \in \mathrm{Hom}(\varphi, \varphi')$ is an isomorphism iff f is invertible in $\mathbb{C}_\infty[\tau]$, i.e. $f \in \mathbb{C}_\infty^\times$. $\longleftrightarrow$ $c \in \mathbb{C}_\infty^\times$ with $c\Lambda_1 = \Lambda_2$

$$f\varphi_a = \varphi_a' f \quad \Leftrightarrow \quad f\varphi_a f^{-1} = \varphi_a'$$

isomorphism $\longleftrightarrow$ conjugation.

We get bijections:

$$\left\{\begin{array}{c}\text{isomorphism classes of rank } r \\ \text{Drinfeld } A\text{-modules over } \mathbb{C}_\infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Homothety classes} \\ \text{of rank } r\ A\text{-lattices} \\ \text{in } \mathbb{C}_\infty\end{array}\right\}$$

when $r=1$

$$\longleftrightarrow \left\{\begin{array}{c}\text{isomorphism classes of rank 1} \\ \text{projective } A\text{-modules}\end{array}\right\}$$

$$\mathbb{Z} \to Cl(A) \quad = \text{class group of } A.$$

"There is only one $\mathbb{G}_m$ over $\mathbb{Z}$ because $h(\mathbb{Z}) = 1$".

<u>Complex Multiplication</u> (Drinfeld):

Let M_A^1 denote the (coarse) moduli space of rank 1 Drinfeld A-modules, then

$$M_A^1 = \operatorname{Spec}(\mathcal{O}_H)$$

where $\mathcal{O}_H$ is the integral closure of A in the Hilbert Class Field H of F: the maximal unramified Abelian extension of F which splits completely at ∞.

$$\operatorname{Gal}(H/F) \cong \operatorname{Pic}(A).$$

Compatible with class field theory.

5.1) Algebraic Theory of Drinfeld Modules

$K/\mathbb{F}_q$ field

<u>Def</u>: A Drinfeld A-module over K is an $\mathbb{F}_q$-alg homomorphism

$$\varphi : A \longrightarrow K[\tau]$$
$$a \longmapsto \varphi_a$$

such that $\varphi(A) \not\subset K$.

Notice: $\gamma : A \longrightarrow K$ is $\mathbb{F}_q$-alg hom
$a \mapsto d\varphi_a$ called the characteristic of φ.

$\ker \gamma$ is a prime ideal in A
1) if $\ker \gamma = (0)$, we say φ has generic characteristic
2) if $\ker \gamma = \mathfrak{p}_0 \neq (0)$, " " " " special characteric

Prop: let $\mathfrak{a} \subset A$ be a non-zero ideal prime to $\mathfrak{p}_0$, then

$$\varphi[\mathfrak{a}] := \{x \in K^{alg} \mid \varphi_a(x) = 0 \ \forall a \in \mathfrak{a}\}$$

$$= \bigcap_{a \in \mathfrak{a}} \ker(\varphi_a \mid K^{alg})$$

$$\cong (A/\mathfrak{a}A)^r \quad \text{for some } r \in \mathbb{N}$$

We call this r the rank of φ.

Morphisms: Let L/K field,

$$\mathrm{Hom}_L(\varphi, \varphi') := \{f \in L[\tau] \mid f\varphi_a = \varphi'_a f \ \forall a \in A\}$$

$$\mathrm{End}_L(\varphi) = \{f \in L[\tau] \mid f\varphi_a = \varphi_a f \ \forall a \in A\}$$

$$= \mathrm{Cent}_{L[\tau]} \varphi(A)$$

If φ has generic char, then φ comes from a lattice in $\mathbb{C}_\infty$

Can use analytic theory. (Lefschetz Principle)

If φ has special char γ with $\mathfrak{P}_0 = \ker \gamma$, then:

- $\mathrm{End}_L(\varphi) \otimes_A F$ is a division algebra over F of dimension $\le r^2$ over F.
- $\varphi[\mathfrak{P}_0] \cong (A/\mathfrak{P}_0)^{r-h}$, $1 \le h \le r$ is called the height of φ.

  If $h = r$, φ is super singular.

6) <u>Galois Theory</u>: φ Drinfeld A-module of rank r over K

$G_K = \mathrm{Gal}(K^{sep}/K)$

$\mathfrak{n} \subset A$ non-zero ideal prime to $\mathfrak{P}_0$ ($= (0)$ or not)

$\varphi[\mathfrak{n}] \cong (A/\mathfrak{n})^r$, $\varphi[\mathfrak{n}] \subset K^{sep}$

(because $n \in \mathfrak{n}$, $n \notin \ker \gamma$, $d\varphi_n \neq 0$
$\Rightarrow \varphi_n$ is separable)

So G_K acts on $\varphi[\mathfrak{n}]$ giving us a Galois representation

$$\rho_{\mathfrak{n}} : G_K \longrightarrow \mathrm{Aut}_{A/\mathfrak{n}}\, \varphi[\mathfrak{n}] \cong GL_r(A/\mathfrak{n})$$

<u>Question</u>: How big is the image?

1) If φ has CM by $R = \mathrm{End}_{\bar{K}}(\varphi) = \mathrm{End}_K(\varphi)$.

Then φ is a rank 1 Drinfeld R-module, so

$\rho_{\mathfrak{n}}$ factors through $GL_1(R/\mathfrak{n}R) = (R/\mathfrak{n}R)^*$

so the Galois groups obtained are abelian

2) If $\mathrm{End}_{\bar{K}}(\varphi) = A$, and K is finitely generated over $\mathbb{F}_q$ then

the index of $\rho_n(G_K)$ in $GL_r(A/n)$ is bounded independently of n.

(Analogue of Serre's Open Image Theorem)

Proved by R. Pink & E. Rütsche (2008)

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7) Special case when $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$.

Now $\varphi : A \to K[\tau]$ is uniquely determined by

$$\varphi_T = \gamma(T) + g_1 \tau + \dots + g_r \tau^r \qquad r = \text{rank}$$

$$\Delta := g_r \neq 0 \qquad \text{discriminant}$$

7.1) <u>$r=1$ case : Carlitz Module (1930's)</u>

$Cl(A) = 1 \Rightarrow$ only one rank 1 Drinfeld $\mathbb{F}_q[T]$-module up to isomorphism over $\bar{F}$.

$$C : A \to F[\tau] \qquad (\gamma : A \subset F)$$
$$T \longmapsto T + \tau$$

is called the Carlitz Module

Let $0 \neq N \in \mathbb{F}_q[T]$, $C[N] = \langle S_N \rangle \cong A/NA$

$F_N := F(\zeta_N)$ cyclotomic extension of F

$$\mathrm{Gal}(F_N/F) \xrightarrow{\sim} (A/NA)^*$$

Splitting of primes in F_N/F works exactly the same way as in $\mathbb{Q}(\zeta_N)/\mathbb{Q}$

$$\begin{array}{lll} F_N & & \\ \Big| \ \mathbb{F}_q^* & \leftarrow & \text{each } \infty_i \text{ totally and tamely ramified} \\ F_N^+ & & \\ \Big| & \leftarrow & \infty \text{ splits completely} \\ F & & \end{array}$$

$$\begin{array}{ll} \mathbb{Q}(\zeta_N) & \\ \Big| \ 2 & \text{totally imag} \\ \mathbb{Q}(\zeta_N)^+ & \\ \Big| & \text{totally real} \\ \mathbb{Q} & \end{array}$$

$\Rightarrow$ F_N/F is purely geometric, only tamely ramified at ∞

Kronecker-Weber Theorem (Hayes):

The maximal abelian extension of F is obtained by adjoining to F:

i) All constants $\overline{\mathbb{F}_q}$

ii) All N-torsion of the Carlitz Module

iii) $(\frac{1}{T})^n$-torsion of the Carlitz Module for $\mathbb{F}_q[\frac{1}{T}]$.

C is associated to the following lattice in C_∞:

$$\bar{\omega} A \subset C_\infty \quad , \text{ where}$$

$\bar{\omega}$ is a transcendental over F $\quad \sim 2\pi i \in \mathbb{C}$

The Carlitz exponential function is:

$$e_{\bar{\omega}A}(z) = \sum_{i=0}^{\infty} \frac{1}{D_i} \tau^i \qquad \sim \left(\exp(z) = \sum_{i=0}^{\infty} \frac{1}{i!} z^i \right)$$

where D_i = product of all monic polynomials in $\mathbb{F}_q[T]$ of degree i.

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