Algebraic Geometry
start up course

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This is a geometric introduction into algebraic geometry. We hope to acquaint the readers with some basic figures underlying the modern algebraic technique and show how to translate things from infinitely rich (but quite intuitive) world of figures to restrictive (in fact, finite) but precise langauge of formulae. Lecture notes are supplied with assessed home tasks and examples of written tests.

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2008

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§1. Projective Spaces.

1.1. Polynomials. Let V be n-dimensional vector space over an arbitrary field $\Bbbk$. Its *dual space* V^* is the space of all $\Bbbk$-linear maps $V \longrightarrow \Bbbk$. Given a basis $e_1, e_2, \dots, e_n$ for V, its *dual basis* for V^* consists of the coordinate forms $x_1, x_2, \dots, x_n$ defined by prescriptions

$$x_i(e_j) = \begin{cases} 1 \text{ , if } i = j \\ 0 \text{ , otherwise} . \end{cases}$$

One can treat each polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ as a function on V whose value at a vector $v \in V$ with coordinates $(v_1, v_2, \dots, v_n)$ w.r.t. the basis $(e_1, e_2, \dots, e_n)$ is equal to $f(v_1, v_2, \dots, v_n)$, i.e. to the result of substitution of values $v_i \in \Bbbk$ instead of the variables x_i. This gives $\Bbbk$-algebra homomorphism

$$\varphi : \Bbbk[x_1, x_2, \dots, x_n] \longrightarrow \{\Bbbk\text{-valued functions on } V\} . \tag{1-1}$$

1.1.1. LEMMA. *The homomorphism* (1-1) *is injective*[1] *iff* $\Bbbk$ *is infinite.*

Proof. If $\Bbbk$ is finite and consists of q elements, then the space of $\Bbbk$-valued functions on V consists of q^{q^n} elements and is finite as well. Since the polynomial algebra is infinite, $\ker \varphi \neq 0$. The inverse argumentation uses the induction on $n = \dim V$. When $n = 1$, any non zero polynomial $f(x)$ has no more than $\deg f$ roots. Thus, $f(x) \equiv 0$ as soon $f(v) = 0$ for infinitely many $v \in V \simeq \Bbbk$. For $n > 1$ we can write a polynomial f as a polynomial in x_n with the coefficients in $\Bbbk[x_1, x_2, \dots, x_{n-1}]$: $f(x_1, x_2, \dots, x_n) = \sum\limits_{\nu} f_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^\nu$. Evaluating all f_ν at an arbitrary point $(v_1, v_2, \dots, v_{n-1}) \in \Bbbk^{n-1}$, we get a polynomial in x_n with constant coefficients and identically zero values. It should be the zero polynomial by the above reason. Hence, each f_ν gives the zero function on $\Bbbk^{n-1}$. By the inductive assumption, all $f_\nu = 0$ as the polynomials. □

1.2. Affine space $\mathbb{A}^n = \mathbb{A}(V)$, of dimension n, is associated with n-dimensional vector space V. The points of $\mathbb{A}(V)$ are the vectors of V. The point corresponding to the zero vector is called *the origin* and is denoted it by O. All other points can be imagined as «the ends» of non zero vectors «drawn» from the origin. The homomorphism (1-1) allows to treat the polynomials as the functions on $\mathbb{A}(V)$. Although this construction does depend on the choice of a basis in V, the resulting space of functions on $\mathbb{A}(V)$, i.e. the image of homomorphism (1-1), does not. It is called an algebra of *polynomial* (or *algebraic*) functions on $\mathbb{A}(V)$. A subset $X \subset \mathbb{A}(V)$ is called *an affine algebraic variety*, if it can be defined by some (maybe infinite) system of polynomial equations.

1.3. Projective space $\mathbb{P}_n = \mathbb{P}(V)$, of dimension n, is associated with a vector space V of dimension $(n+1)$. By the definition, the points of $\mathbb{P}(V)$ are 1-dimensional vector subspaces in V or, equivalently, the lines in $\mathbb{A}^{n+1} = \mathbb{A}(V)$ passing through the origin. To see them as «the usual points» one should use a screen, i.e. some affine hyperplane of codimension one $U \subset \mathbb{A}(V)$, which does not contain the origin like on fig 1◇1. Such a screen is called *an affine chart* on $\mathbb{P}(V)$. Of course, no affine chart does cover the whole of $\mathbb{P}(V)$. The difference $U_\infty \overset{\text{def}}{=} \mathbb{P}_n \setminus U$ consists of all lines lying in a parallel copy of U drawn through O. It is naturally identified with $\mathbb{P}_{n-1} = \mathbb{P}(U)$. Thus, $\mathbb{P}_n = U \sqcup U_\infty = \mathbb{A}^n \sqcup \mathbb{P}_{n-1}$. Iterating this decomposition, one can split $\mathbb{P}_n$ into disjoint union of affine spaces: $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \dots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$.

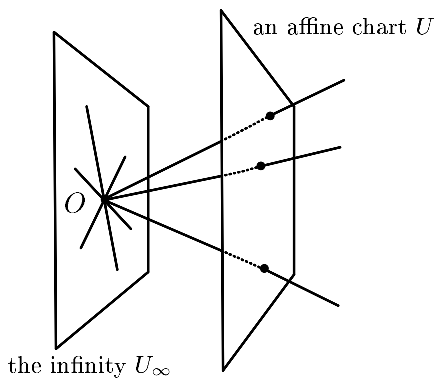


Fig 1◇1. The projective world.

1.4. Global homogeneous coordinates. Let us fix a basis for V and use it to write vectors $x \in V$ as the coordinate rows $x = (x_0, x_1, \dots, x_n)$. Two vectors $x, y \in V$ represent the same point $p \in \mathbb{P}(V)$ iff they are proportional, i.e. $x_\nu = \lambda y_\nu$ for all $\nu = 0, 1, \dots, n$ and some non zero $\lambda \in \Bbbk$.

[1] i. e. different polynomials always give the different polynomial functions on V

Thus, the point $p \in \mathbb{P}(V)$ can be coordinated by the collection of ratios $(x_0 : x_1 : \ \ldots \ : x_n)$. This ratios are called *homogeneous coordinates* on $\mathbb{P}(V)$ w. r. t. the chosen basis of V.

Since we have usually $f(x) \neq f(\lambda x)$ for a non zero polynomial $f \in \Bbbk[x_0, x_1, \ldots, x_n]$, the polynomials do not produce the functions on $\mathbb{P}(V)$ any more. But if f is a homogeneous polynomial, say of degree $d > 0$, then its zero set $(f)_0 \stackrel{\text{def}}{=} \{ v \in V \mid f(v) = 0 \}$ is well defined in $\mathbb{P}(V)$, because $f(x) = 0 \iff f(\lambda x) = \lambda^d f(x) = 0$. This zero set is called *a projective hypersurface* of degree d. The intersections of such hypersurfaces[1] are called *projective algebraic varieties*.

For example, the equation $x_0^2 + x_1^2 = x_2^2$ defines a curve $C \subset \mathbb{P}_2$. When $\operatorname{char}(\Bbbk) \neq 2$, this curve is called *non degenerate plane conic*.

We write $S^d(V^*) \subset \Bbbk[x_0, x_1, \ldots, x_n]$ for the subspace of all homogeneous polynomials of degree d. Note that as a vector space over $\Bbbk$ the polynomial algebra splits into the direct sum

$$\Bbbk[x_0, x_1, \ldots, x_n] = \bigoplus_{d \geqslant 0} S^d(V^*) \ , \quad \text{and} \quad S^k(V^*) \cdot S^\ell(V^*) \subset S^{k+\ell}(V^*) \ ,$$

i. e. $\Bbbk[x_0, x_1, \ldots, x_n]$ is a *graded algebra* with graded components $S^d(V^*)$. Since proportional equations define the same hypersurfaces, the hypersurfaces $S \subset \mathbb{P}(V)$ of degree d correspond to the points of the projective space $\mathbb{P}(S^d(V^*))$.

1.5. Local affine coordinates. Any affine chart $U \subset \mathbb{A}(V)$ can by uniquely given by the equation $\alpha(x) = 1$, where $\alpha \in V^*$. We will write U_α for this chart. One dimensional subspace spanned by $v \in V$ is visible in chart U_α iff $\alpha(v) \neq 0$. A point that represents this subspace in U_α is $v/\alpha(v) \in U$. If fix some n linear forms $\xi_1, \xi_2, \ldots, \xi_n \in V^*$ such that $n+1$ forms $\alpha, \xi_1, \xi_2, \ldots, \xi_n$ form a basis of V^*, we can use their restrictions onto U as *local affine coordinates* inside $U_\alpha \subset \mathbb{P}_n$. In terms of these coordinates, a point $p \in \mathbb{P}_n$ corresponding to $v \in V$ is coordinated by n numbers $\xi_i(v/\alpha(v))$, $1 \leqslant i \leqslant n$. These coordinates depend on α and the choice of ξ_i's. Note that they are rational *linear fractional* functions of the homogeneous coordinates and a sentence «p is running away from U_α to infinity» means nothing but «$\alpha(p) \to 0$», which leads to unbounded increasing of the local affine coordinates.

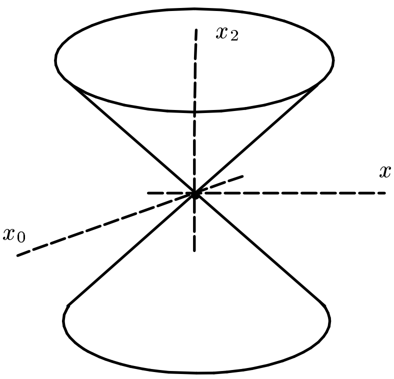


Fig 1◇2. The cone.

1.5.1. Example: affine conics. Let us consider local equations for the plane conic

$$x_0^2 + x_1^2 = x_2^2 \tag{1-2}$$

in some affine charts. In the chart U_{x_0}, given by the equation $\{x_0 = 1\}$, we can chose local affine coordinates $t_1 = x_1|_{U_0} = x_1/x_0$ and $t_2 = x_2|_{U_{x_0}} = x_2/x_0$. Dividing the both sides of (1-2) by x_2^2, we get for $C \cap U_0$ the equation $t_2^2 - t_1^2 = 1$, i. e. $C \cap U_{x_0}$ is a hyperbola. Similarly, in a chart $U_{x_2} = \{x_0 = 1\}$ with local affine coordinates $t_0 = x_0/x_2$, $t_1 = x_1/x_2$ we get the equation $t_0^2 + t_1^2 = 1$, i. e. $C \cap U_{x_2}$ is a circle. Finally, consider a chart $U_{x_2 - x_1}$ given by $x_2 - x_1 = 1$ with local affine coordinates $t_0 = x_0|_{U_{x_2-x_1}} = \dfrac{x_0}{x_2 - x_1}$, $t_1 = (x_2 + x_1)|_{U_{x_2-x_1}} = \dfrac{x_2 + x_1}{x_2 - x_1}$. After dividing by $(x_2 - x_1)^2$ and some eliminations, we see that $C \cap U_{x_2 - x_1}$ is the parabola $t_1 = t_0^2$.

Exercise 1.1. The affine cone $x_0^2 + x_1^2 = x_2^2$ in $\mathbb{A}^3$ is drawn on fig 1◇2. Picture there each of 3 previous affine charts and outline their intersections with the cone.

1.6. Projective closure. Any affine algebraic variety $X \subset \mathbb{A}^n$ is always an affine piece of projective algebraic variety $\widetilde{X} \subset \mathbb{P}_n$ called *a projective closure of* X. Indeed, if X is given by polynomial equations $\{ f_\nu(t_1, t_2, \ldots, t_n) = 0 \}$, we substitute $t_i = x_i/x_0$ and multiply the ν-th equation by $x_0^{\deg f_\nu}$. Then the resulting equations $\widetilde{f}_\nu(x_0, x_1, \ldots, x_n) = 0$ become homogeneous and define a projective algebraic variety $\widetilde{X} \subset \mathbb{P}_n$ such that $\widetilde{X} \cap U_{x_0} = X$, where U_{x_0} is the affine chart $x_0 = 1$. Geometrically, $\widetilde{X}$ is the union of X with all its *asymptotic directions*. Thus, the projective langauge allows to treat the affine asymptotic directions as ordinary points lying at infinity.

1.7. Standard affine covering and gluing rules. Clearly, the whole of $\mathbb{P}_n$ is covered by $(n+1)$ affine charts $U_\nu \stackrel{\text{def}}{=} U_{x_\nu}$ given in $\mathbb{A}^{n+1}$ by equations $x_\nu = 1$. This covering is called *the standard affine covering*.

[1] maybe infinite families of hypersurfaces of different degrees

For each $\nu = 0, 1, \dots, n$ we take n forms

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}\,, \quad \text{where } 0 \leqslant i \leqslant n\,,\ i \neq \nu$$

as the standard local affine coordinates on U_ν. Topologically, this means that $\mathbb{P}_n$ is constructed from $(n+1)$ distinct copies of $\mathbb{A}^n$ denoted as $U_0, U_1, \dots, U_n$ by gluing them together along the actual intersections $U_\mu \cap U_\nu \subset \mathbb{P}_n$ (i. e. a point of U_ν is identified with a point of U_μ under this gluing rules iff they correspond to the same point of $\mathbb{P}_n$). In term of the homogeneous coordinates, the intersection $U_\mu \cap U_\nu$ consists of all x such that both x_μ and x_ν are non zero. This locus is presented inside U_μ and U_ν by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$ respectively. Thus a point $t^{(\mu)} \in U_\mu$ is glued with a point $t^{(\nu)} \in U_\nu$ iff

$$t_\nu^{(\mu)} = 1/t_\mu^{(\nu)} \qquad \text{and} \qquad t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)} \quad \text{for} \quad i \neq \mu, \nu\,.$$

The right hand sides of these equations are called *the transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$ over $U_\mu \cap U_\nu$.

For example, $\mathbb{P}_1$ can be produced from two copies of $\mathbb{A}^1$ by identifying the point t in one of them with the point $1/t$ in the other for all $t \neq 0$.

Exercise 1.2*. If you have some experience in smooth topology, prove that real and complex projective lines are analytic manifolds isomorphic to the circle S^1 (in real case) and to the Riemann sphere S^2 (in complex case).

1.8. Projective subspaces. A closed projective algebraic subset is called *a projective subspace* if it can be given by a system of *linear* homogeneous equations. Any projective subspace $L \subset \mathbb{P}(V)$ has a form $L = \mathbb{P}(W)$, where $W \subset V$ is a vector subspace. Note that 0-dimensional projective subspaces[1] coincide with the points. Since $\operatorname{codim}_{\mathbb{P}(V)}\mathbb{P}(W) = \operatorname{codim}_V W$, we have $L_1 \cap L_2 \neq \varnothing$ for any two projective subspaces L_1 and L_2 such that $\operatorname{codim} L_1 + \operatorname{codim} L_2 \leqslant n$. For example, any two lines on $\mathbb{P}_2$ have non empty intersection[2].

Two projective subspaces L_1 and L_2 in $\mathbb{P}_n$ are called *complementary* to each other, if

$$L_1 \cap L_2 = \varnothing \qquad \text{and} \qquad \dim L_1 + \dim L_2 = n - 1\,.$$

For example, any two skew lines in 3-dimensional space are complementary.

Exercise 1.3. Show that $\mathbb{P}(U)$ and $\mathbb{P}(W)$ are complementary in $\mathbb{P}(V)$ iff $V = U \oplus W$.

1.8.1. LEMMA. *If $L_1, L_2 \subset \mathbb{P}(V)$ are two complementary linear subspaces, then any point $p \in \mathbb{P}(V) \setminus (L_1 \cup L_2)$ lies on a unique line crossing the both subspaces.*

Proof. We have $V = U_1 \oplus U_2$, where $\mathbb{P}(U_i) = L_i$. So, any $v \in V$ has *a unique* decomposition $v = u_1 + u_2$ with $u_i \in U_i$. If $v \notin U_1 \cup U_2$, then both u_i, u_2 are non zero and span *a unique* 2-dimensional subspace that contains v and has non zero intersections with both U_i. □

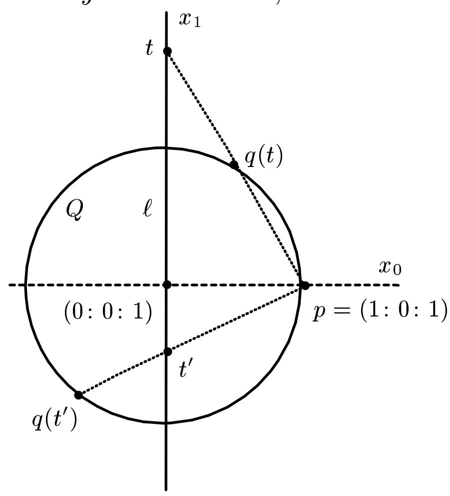


Fig 1◇3. Projecting a conic.

1.9. Projections. For any two complementary projective subspaces $L_1, L_2 \subset \mathbb{P}_n$, a *projection* from L_1 onto L_2 is a map $\pi_{L_2}^{L_1} : (\mathbb{P}_n \setminus L_1) \longrightarrow L_2$ that sends any point $q \in L_2$ to itself and any point $p \in \mathbb{P}_n \setminus (L_1 \sqcup L_2)$ to $\ell \cap L_2$, where ℓ is the unique line passing through p and crossing both L_1 and L_2 in accordance with n° 1.8.1. In homogeneous coordinates $(x_0 : x_1 : \dots : x_n)$ such that L_1 is coordinated by $(x_0 : x_1 : \dots : x_m)$ and L_2 is coordinated by $(x_{m+1} : x_{m+2} : \dots : x_n)$, the projection $\pi_{L_2}^{L_1}$ is nothing but taking $x_\nu = 0$ for $0 \leqslant \nu \leqslant (m+1)$.

1.9.1. Example: projecting a conic onto a line. Consider the projection $\pi_\ell^p : Q \longrightarrow \ell$ of the plane conic (1-2) onto the line $\ell = \{x_0 = 0\}$ from the point $p = (1 : 0 : 1) \in Q$. Inside the standard affine chart U_2, where $x_2 = 1$, it looks like on fig 1◇3. It is bijective, because the pencil of all lines passing through p is bijectively parameterized

[1] that is, $\mathbb{P}_0 = \mathbb{P}(k^1)$

[2] in terms of $\mathbb{A}^3$ this means that any two planes containing the origin are intersected along a line

by the points $t \in \ell$ and any such a line (pt) intersects Q exactly in one more point $q = q(t)$ in addition to p except for the the tangent line at p, which is given by $x_0 = x_2$ and crosses ℓ at the point[1] $t = (0 : 1 : 0)$ corresponding to $q(t) = p$ itself. Moreover, this bijection is *birational*, i. e. the corresponding $(q_0 : q_1 : q_2) \in Q$ and $(0 : t_1 : t_2) \in L$ are *rational* algebraic functions of each other. Namely, $(t_1 : t_2) = (\, q_1 : (q_2 - q_0)\,)$ and $(q_0 : q_1 : q_2) = (\,(t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)\,)$.

Exercise 1.4. Check these formulas and note that while (t_1, t_2) runs through $\mathbb{Z} \times \mathbb{Z}$ the second formula gives the full list of the pythagorian triples $(q_0 : q_1 : q_2)$ (i.e. all the right triangles with integer side lengths).

1.10. Matrix notations for linear maps. Let $\mathrm{Hom}(U, W)$ be the space of all $\Bbbk$-linear maps from n-dimensional vector space U to m-dimensional vector space W. Denote by $\mathrm{Mat}_{m\times n}(\Bbbk)$ the space of matrices with m rows, n columns, and entries in the field $\Bbbk$. Any pair of basises $\{u_1, u_2, \dots, u_n\} \subset U$ and $\{w_1, w_2, \dots, w_m\} \subset W$ presents an isomorphism $\mathrm{Hom}(U, W) \xrightarrow{\sim} \mathrm{Mat}_{m\times n}(\Bbbk)$ that sends an operator $U \xrightarrow{\alpha} W$ to a matrix $A = (a_{ij})$ whose j-th column consists of m coordinates of the vector $\alpha(u_j) \in W$ w.r.t. the basis $\{w_1, w_2, \dots, w_m\}$, i.e. $\alpha(u_j) = \sum_{\nu=1}^{m} a_{\nu j}\, w_\nu$ or, using the matrix multiplication,

$$(\alpha(u_1),\, \alpha(u_2),\, \dots\, ,\, \alpha(u_n)) = (w_1, w_2, \dots, w_m) \cdot A\,.$$

Let us write ${}^t x$ and ${}^t y$ for the columns obtained by transposing coordinate rows of $x = (x_1, x_2, \dots, x_m) \in U$ and $y = (y_1, y_2, \dots, y_n) = \alpha(x) \in W$. Then

$$\begin{aligned}(w_1, w_2, \dots, w_m) \cdot {}^t y = \alpha(x) = \alpha\big((u_1, u_2, \dots, u_n) \cdot {}^t x\big) = \\ = \big(\alpha(u_1),\, \alpha(u_2),\, \dots\, ,\, \alpha(u_n)\big) \cdot {}^t x = (w_1, w_2, \dots, w_m) \cdot A \cdot {}^t x\end{aligned}$$

implies that ${}^t y = A \cdot {}^t x$.

1.11. Linear projective transformations. If $\dim U = \dim W = (n+1)$, then any linear isomorphism $U \xrightarrow{\alpha} W$ induces the bijection $\mathbb{P}(U) \xrightarrow{\overline{\alpha}} \mathbb{P}(W)$, which is called *the projective linear transformation* or *the linear isomorphism*. A point set $\{p_1, p_2, \dots, p_m\} \subset \mathbb{P}_n$ is called *linearly general*, if any $(n+1)$ of p_i don't lay together in any hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$. Equivalently, the points $\{p_i\}$ are linearly general in $\mathbb{P}_n = \mathbb{P}(V)$ iff $(n+1)$ vectors representing any $(n+1)$ of them always form a basis of V.

1.11.1. LEMMA. *For any two linearly general collections of $(n+2)$ points $\{p_0, p_1, \dots, p_{n+1}\} \in \mathbb{P}(U)$ and $\{q_0, q_1, \dots, q_{n+1}\} \in \mathbb{P}(V)$, where $\dim U = \dim V = (n+1)$, there exists a unique up to proportionality linear isomorphism $V \xrightarrow{\alpha} W$ such that $\overline{\alpha}(p_i) = q_i$ $\forall i$. In particular, two matrices give the same projective linear transformation iff they are proportional.*

Proof. Fix some vectors u_i and w_i representing the points p_i and q_i. By the linear generality, we can take $\{u_1, u_2, \dots, u_{n+1}\}$ and $\{w_1, w_2, \dots, w_{n+1}\}$ as the basises in U and W and identify a map $\alpha \in \mathrm{Hom}(U, W)$ by the square matrix in these basises. Then, $\overline{\alpha}(p_i) = q_i$ for $1 \leqslant i \leqslant (n+1)$ iff the matrix A of α is diagonal, say with $(d_1, d_2, \dots, d_{n+1})$ on the main diagonal. Now consider the first vectors u_0 and w_0. Again by the linear generality, all the coordinates of $u_0 = (x_1, x_2, \dots, x_{n+1})$ w.r.t. the basis $\{u_i\}_{1\leqslant i\leqslant(n+1)}$ and all the coordinates of $w_0 = (y_1, y_2, \dots, y_{n+1})$ w.r.t. the basis $\{w_i\}_{1\leqslant i\leqslant(n+1)}$ are non zero. Since the coordinates of $\alpha(u_0)$ are proportional to the ones of w_0, we have $y_i : y_j = (d_i x_i) : (d_j x_j)$ $\forall\, i, j$. Hence, all d_i are uniquely recovered from just one of them, say d_1, as $d_i = d_1 \cdot (y_i x_1) : (x_i y_1)$. $\square$

Exercise 1.5. Let ℓ_1 and ℓ_2 be two lines on $\mathbb{P}_2$. Fix any point p outside $\ell_1 \cup \ell_2$ and consider projective linear isomorphism $\ell_1 \xrightarrow{\gamma_p} \ell_2$ that sends $t \in \ell_1$ to the intersection point $(tp) \cap \ell_2$. Check that γ_p is a linear isomorphism.

1.12. Linear projective group. All linear isomorphisms $V \longrightarrow V$ form a group denoted by $\mathrm{GL}(V)$. It acts on $\mathbb{P}(V)$. By n° 1.11.1, the kernel of this action coincides with the subgroup of all scalar dilatations $H \subset \mathrm{GL}(V)$. Hence, the group of all projective linear automorphisms of $\mathbb{P}(V)$ is equal to the factor group $\mathrm{GL}(V)/H$, which is denoted by $\mathrm{PGL}(V)$ and called the *projective linear group*. Fixing a basis $\{e_0, e_1, \dots, e_n\} \subset V$, we can identify $\mathrm{GL}(V)$ with the group $\mathrm{GL}_{n+1}(\Bbbk) \subset \mathrm{Mat}_{n+1}(\Bbbk)$ of all non

[1] this intersection point lies at the infinity on fig 1◇3

degenerated square matrices. Under this identification the dilatations go to the scalar diagonal matrices and $\mathrm{PGL}(V)$ turns into the group $\mathrm{PGL}_{n+1}(\Bbbk) \stackrel{\text{def}}{=} \mathrm{GL}_{n+1}(\Bbbk)/\{\text{scalar diagonal matrices } \lambda E\}$, of all non degenerate square matrices considered up to proportionality.

1.12.1. Example: linear fractional group and cross-ratio. $\mathrm{PGL}_2(\Bbbk)$ consists of all 2×2 - matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$ considered up to proportionality. It acts on $\mathbb{P}_1$ via $(x_0 : x_1) \longmapsto ((ax_0 + bx_1) : (cx_0 + dx_1))$. In the standard affine chart $U_0 \simeq \mathbb{A}^1$ with affine coordinate $t = x_1/x_0$ this action looks like the linear fractional transformation $t \longmapsto (ct + d)/(at + b)$.

Exercise 1.6. Verify by the straightforward computation that $(AB)(t) = A(B(t))$.

Theorem n° 1.11.1 says that for any 3 different points p, q, r there exists a unique linear fractional transformation α such that $\alpha(p) = 0$, $\alpha(q) = 1$, and $\alpha(r) = \infty$. This is clear, because $p \mapsto 0$ and $r \mapsto \infty$ force such a transformation to take $t \xmapsto{\alpha} \vartheta \cdot (t - p)/(t - r)$, where $\vartheta \in \Bbbk$. Substituting $t = q$, we get $\vartheta = (q - r)/(q - p)$, i. e. the required transformation is

$$t \longmapsto \frac{q-r}{q-p} \cdot \frac{t-p}{t-r} .$$

The right hand site is called the *cross-ratio* of 4 points t, p, q, r on $\mathbb{P}_1$.

Exercise 1.7. Show that the cross-ratio does not depend on choice of coordinates and is invariant under the action of PGL_2 on the quadruples of points.

1.12.2. PROPOSITION. *If a bijective mapping*

$$\mathbb{P}_1 \setminus \{\textit{finite collection of points}\} \xrightarrow{\varphi} \mathbb{P}_1 \setminus \{\textit{finite collection of points}\}$$

can be given by a formula $\varphi(x_0 : x_1) = (f_0(x_0, x_1) : f_1(x_0, x_1))$, where f_i are rational algebraic functions, then φ has to be a linear fractional transformation.

Proof. Multiplying $(f_0 : f_1)$ by the common denominator and eliminating common factors we can assume that f_i are *coprime polynomials*. To produce a well defined map, they have to be homogeneous of the same positive degree d. Since φ is bijective, each $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1 \setminus \{\text{finite collection of points}\}$ has precisely one preimage. This means that for infinitely many values of ϑ the homogeneous equation

$$\vartheta_1 \cdot f_0(x_0, x_1) - \vartheta_0 \cdot f_1(x_0, x_1) = 0 \tag{1-3}$$

has just one root up to proportionality, i. e. its left hand side is a pure d-th power of some linear form in $(x_0 : x_1)$.

All homogeneous polynomials of degree d in $(x_0 : x_1)$ considered up to a scalar factor form the projective space $\mathbb{P}_d = \mathbb{P}(S^d U^*)$, where U is the 2-dimensional vector subspace underlying $\mathbb{P}_1$ in question. When ϑ varies through $\mathbb{P}_1$, the equations (1-3) draw a straight line $(f_0 f_1)$ inside this $\mathbb{P}_d$ whereas pure d-th powers of linear forms form there some *twisted* curve, which is called *the Veronese curve* of degree d. Lemma n° 1.12.3 below implies that for $d \geqslant 2$ any 3 points on the Veronese curve are non collinear. Since in our case an infinite set of points on the line (1-3) lies on the Veronese curve, we conclude that $d = 1$, i. e. φ is a projective linear isomorphism. □

1.12.3. LEMMA. *Let us define the Veronese curve of degree d as an image of the Veronese map*

$$\mathbb{P}_1 = \mathbb{P}(U^*) \xrightarrow{v_d} \mathbb{P}_d = \mathbb{P}\left(S^d(U^*)\right) \tag{1-4}$$

that takes a linear form $\psi \in U^*$ to its d-th power $\psi^d \in S^d(U^*)$. If the ground field $\Bbbk$ contains more than d elements, then for each $k = 2, 3, \ldots, d$ any $(k+1)$ distinct points of the Veronese curve can not belong to the same $(k-1)$-dimensional projective subspace.

Proof. Let us write $\psi \in U^*$ and $f \in S^d(U^*)$ as $\psi = \alpha_0 x_0 + \alpha_1 x_1$, $f = \sum\limits_{\nu=0}^{d} a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu$ and use $(\alpha_0 : \alpha_1)$, $(a_0 : a_1 : \ldots : a_d)$ as homogeneous coordinates on $\mathbb{P}_1$ and $\mathbb{P}_d$ respectively. It is enough to verify the case $k = d$, which implies all the other cases. Consider the intersection of the Veronese curve with $(d-1)$-dimensional projective hyperplane given by a linear equation $\sum A_\nu a_\nu = 0$. Its preimage under the Veronese map (1-4) consists of all $(\alpha_0 : \alpha_1) \in \mathbb{P}_1$ satisfying non trivial homogeneous equation $\sum A_\nu \cdot \alpha_0^{d-\nu} \alpha_1^\nu = 0$ of degree d. Up to proportionality, it has at most $d + 1$ distinct roots. □

§2. Projective quadrics.

In §2 we will assume that $\operatorname{char}\Bbbk \ne 2$.

2.1. Quadratic and bilinear forms. A zero set $Q \subset \mathbb{P}(V)$ of non zero quadratic form $q \in S^2V^*$ is called *a projective quadric*. If $2 \ne 0$ in $\Bbbk$, then the explicit expression for q in homogeneous coordinates can be written as

$$q(x) = \sum_{i,j} a_{ij}\, x_i x_j = x \cdot A \cdot {}^t\!x \;,$$

where $x = (x_0, x_1, \dots, x_n)$ is the coordinate row, ${}^t\!x$ is its transposed column version, and $A = (a_{ij})$ is a *symmetric* matrix over $\Bbbk$, whose non-diagonal element $a_{ij} = a_{ji}$ equals one half of the coefficient at $x_i x_j$ in q. This matrix is called *the Gram matrix* of q. In other words, there exists a unique bilinear form $\widetilde{q}(u,w)$ on $V \times V$ such that $q(x) = \widetilde{q}(x,x)$. This form is called *the polarization* of q. It can be expressed through q in the following pairwise equivalent ways:

$$\widetilde{q}(x,y) = \sum_{ij} a_{ij}\, x_i y_j = x \cdot A \cdot {}^t\!y = \frac{1}{2}\sum_i y_i \frac{\partial q(x)}{\partial x_i} = \frac{q(x+y) - q(x) - q(y)}{2} = \frac{q(x+y) - q(x-y)}{4}$$

Note that $\widetilde{q}$ can be treated as a kind of scalar product on V. Then the elements of the Gram matrix become the scalar products of basic vectors: $a_{ij} = \widetilde{q}(e_i, e_j)$. Thus, tacking another basis

$$(e_0', e_1', \dots, e_n') = (e_0, e_1, \dots, e_n) \cdot C \;,$$

we change the Gram matrix by the rule $A \longmapsto A' = {}^t C \cdot A \cdot C$.

Note that under changes of basis the determinant of the Gram matrix *the Gram determinant* $\det q \overset{\text{def}}{=} \det A$ is multiplied by non zero square from $\Bbbk^*$. Thus, its class modulo multiplication by non zero squares $[\det q] \in \Bbbk/\Bbbk^2$ does not depend on a choice of basis. Two quadrics are called *isomorphic* or *projectively equivalent*, if their equations can be transformed to each other by a linear change of basis. A quadric is called *smooth*, if $\det q \ne 0$. Otherwise, it is called *singular*. We see that projective equivalence preserves smoothness and the class of $\det q$ in $\Bbbk^*/\Bbbk^{*2}$, where $\Bbbk^*$ is the multiplicative group of $\Bbbk$.

2.1.1. PROPOSITION (LAGRANGE THEOREM). *The Gram matrix of any quadratic form q on V can be diagonalized by appropriate change of basis in V.*

Proof. If $q \equiv 0$, its Gram matrix is already diagonal. If not, then $q(v) = \widetilde{q}(v,v) \ne 0$ for some $v \in V$. Take this v as the first vector of the basis being constructed. Note that any $u \in V$ is uniquely decomposed as $u = \lambda v + w$ with $\lambda \in \Bbbk$ and $w \in v^\perp = \{\, w \in V \mid \widetilde{q}(v,w) = 0 \,\}$. Indeed, the only possibility is $\lambda = \widetilde{q}(v,u)/\widetilde{q}(v,v)$, $w = u - (\widetilde{q}(v,u)/\widetilde{q}(v,v)) \cdot v$ and it works. Thus, $V = \Bbbk \cdot v \oplus v^\perp$ and we can repeat the arguments to $v^\perp \not\subseteq V$ instead of V e. t. c. $\square$

2.1.2. COROLLARY. *If $\Bbbk$ is algebraically closed, then any quadric can be defined in appropriate coordinates by an equation of the form $\sum x_i^2 = 0$. In particular, all non singular quadrics are projectively equivalent.*

Proof. Diagonal elements of the Gram matrix become units after the change $e_i \longmapsto e_i/\sqrt{q(e_i)}$. $\square$

2.1.3. Example: quadrics on $\mathbb{P}_1$ in appropriate coordinates are given either by an equation $a\,x_0^2 + b\,x_1^2 = 0$ or by an equation $a\,x_0^2 = 0$. The second quadric is called *a double point*, because it consists of just one point $(0:1)$, which has «multiplicity 2» in any reasonable sense. Clearly, it is singular (i. e. $\det q = 0$). The first quadric is smooth (i. e. $\det q \in \Bbbk^*$) and either consists of two distinct points or is empty. More precisely, if $-\det q = -ab = \delta^2$ is a square in $\Bbbk^*$, then $Q = \{\, (-\delta : a)\,, (\delta : a) \,\}$. But if $-b/a \equiv -\det q \,(\operatorname{mod} \Bbbk^{*2})$ is not a square, then evidently $Q = \varnothing$. Note that the latter case is impossible when $\Bbbk$ is algebraically closed.

2.2. Quadric and line. It follows from the above example that there are precisely four positional relationships of a quadric Q with a line ℓ: either $\ell \subset Q$, or $\ell \cap Q$ consists of 2 distinct points, or $\ell \cap Q$ is a double point, or $\ell \cap Q = \varnothing$. Moreover, the latter case is impossible when $\Bbbk$ is algebraically closed.

A line ℓ is called *a tangent line* to a quadric Q, if ℓ either lies on Q or crosses Q via a double point.

2.3. Correlations. Any quadratic form q on V induces the linear map $V \xrightarrow{\widehat{q}} V^*$ that sends a vector $v \in V$ to the linear form

$$\widehat{q}(v) : w \longmapsto \widetilde{q}(w,v)$$

The map $\widehat{q}$ is called *the correlation* (or *the polarity*) of the quadratic form q. The matrix of $\widehat{q}$ written in dual bases $\{e_i\} \subset V$, $\{x_i\} \subset V^*$ coincides with the Gram matrix A. In particular, q is smooth iff $\widehat{q}$ is an isomorphism. The space

$$\ker(q) \stackrel{\text{def}}{=} \ker \widehat{q} = \{ v \in V \mid \widetilde{q}(w,v) = 0 \ \forall w \in V \}$$

is called *the kernel* of q. Its projectivization $\operatorname{Sing} Q \stackrel{\text{def}}{=} \mathbb{P}(\ker q) \subset \mathbb{P}(V)$ is called *a vertex space* of Q and $\operatorname{codim}_{\mathbb{P}(V)} \operatorname{Sing} Q$ is called *a corank* of Q.

2.3.1. THEOREM. *The intersection $Q' = L \cap Q$ is non singular for any projective subspace $L \subset \mathbb{P}(V)$ complementary to* $\operatorname{Sing} Q$*; moreover, Q is the cone over Q' with the vertex space* $\operatorname{Sing} Q$*, i.e. Q is the union of all lines crossing both Q' and* $\operatorname{Sing} Q$.

Proof. Take any direct decomposition $V = \ker q \oplus U$ and let $L = \mathbb{P}(U)$. If $u \in U$ satisfy $\widetilde{q}(u,u') = 0 \quad \forall u' \in U$, then automatically $\widetilde{q}(u,v) = 0 \quad \forall v \in V$ and $u = 0$, because of $\ker q \cap U = 0$. Since $Q' = Q \cap L$ is given by the restriction $q|_U$, it is non singular. Further, for any line $\ell = \mathbb{P}(W)$ such that $\dim W \cap \ker Q = 1$ we have $\dim W \cap U = 1$ and $\operatorname{cork} q|_U \geqslant 1$. So, if $\ell \cap \operatorname{Sing} Q = \{p\}$ is just one point, then $\ell \cap L \neq \varnothing$ and either $\ell \subset Q$ or $\ell \cap Q\{p\}$. That's all we need. □

2.3.2. COROLLARY. *A quadric $Q \subset \mathbb{P}_n$ over an algebraically closed field is uniquely up to an isomorphism defined by its corank, which can be equal to* $0, 1, \dots, n$.

Proof. Corank is the number of diagonal zeros in the diagonal Gram matrix. □

2.4. Tangent space T_pQ to a quadric Q at a point $p \in Q$ is defined as the union of all tangent lines passing through p.

2.4.1. LEMMA. *Let p and p' be distinct points and $p \in Q = (q)_0$. The line $\ell = (p\,p')$ is tangent to Q iff $\widetilde{q}(p,p') = 0$, i.e. iff p and p' are orthogonal with respect to polarization of q.*

Proof. Take some vectors u, u' representing p and p'. Then $\ell = \mathbb{P}(U)$. The restriction $q|_U$ has the Gram matrix

$$\begin{pmatrix} \widetilde{q}(u,u) & \widetilde{q}(u,u') \\ \widetilde{q}(u',u) & \widetilde{q}(u',u') \end{pmatrix} .$$

It is singular iff $\widetilde{q}(u,u') = 0$, because of $\widetilde{q}(u,u) = 0$ by the lemma assumption. □

2.4.2. COROLLARY. $p \in \operatorname{Sing} Q \iff$ *T_pQ is the whole space* $\iff \dfrac{\partial q}{\partial x_i}(p) = 0 \ \forall i$. □

2.4.3. COROLLARY. *If $p \in (Q \setminus \operatorname{Sing} Q)$, then $T_pQ = \{x \in \mathbb{P}_n \mid \widetilde{q}(p,x) = 0\}$ is a hyperplane of codimension one.* □

2.4.4. COROLLARY. *Let $p \notin Q$ and a hyperplane $C \subset \mathbb{P}_n$ be given by the equation $\widetilde{q}(p,x) = 0$ in x. Then $Q \cap L$ consists of all points where Q is touched by the tangent lines coming from p.* □

2.5. Polar mappings. The spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are called *dual* and denoted by $\mathbb{P}_n$ and $\mathbb{P}_n^\times$ when a nature of V is not essential. Since any codimension 1 subspace $U \subset V$ is defined by linear form $\xi \in V^*$, which is unique up to proportionality, $\mathbb{P}_n^\times$ is nothing but *the space of hyperplanes* in $\mathbb{P}_n$ and vice versa. If $Q = (q)_0 \subset \mathbb{P}(V)$ is non singular, then the linear isomorphism $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ induced by the correlation $\widehat{q}$ is called *a polarity* of Q. It sends a point $p \in \mathbb{P}_n$ to the hyperplane $L \subset \mathbb{P}_n$ given by the equation $\widetilde{q}(p,x) = 0$ like in the previous Corollary. L is called *a polar* of p and p is called *a pole* of L with respect to q. So, Q is just the set of all points lying on their own polars. Note that some non singular quadratic forms q can produce *empty* quadrics Q over non closed fields but their polar mappings $\widehat{q}$ are always visible.

Exercise 2.1. Show that p lies on the polar of q iff q lies on the polar of p (for any pair of distinct points and any polarity).

Exercise 2.2. Consider a circle in the real Euclidean affine plane $\mathbb{R}^2$. How to draw the polar of a point that lies: a) outside b) inside this circle? Describe geometrically the polarity defined by the «imaginary» circle «given» in $\mathbb{R}^2$ by the equation $x^2 + y^2 = -1$.

2.5.1. PROPOSITION. *Two polarities coincide iff the corresponding quadratic forms are proportional.*

Proof. This follows from n° 1.11.1 □

2.5.2. COROLLARY. *Over an algebraically closed field two quadrics coincide iff their quadratic equations are proportional.*

Proof. Let $Q = Q'$. We can suppose that the quadrics are non singular, because their equations are not changed under direct summation with the kernel $\ker q = \ker q'$. Non singular case is covered by the above proposition. □

2.6. The space of quadrics. All the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ are one-to-one parameterized by the points of the projective space

$$\mathbb{P}_{\frac{n(n+3)}{2}} = \mathbb{P}(S^2V^*) ,$$

which will be referred as *a space of quadrics*. Given a point $p \in \mathbb{P}(V)$, the condition $q(p) = 0$ is a linear condition on $q \in \mathbb{P}(S^2V^*)$, i. e. all quadrics passing through a given point p form a projective hyperplane in the space of quadrics. Since any $n(n+3)/2$ hyperplanes in $\mathbb{P}_{n(n+3)/2}$ have non empty intersection, we come to the following quite helpful conclusion

2.6.1. CLAIM. *Any collection of $n(n+3)/2$ points in $\mathbb{P}_n$ lies on some quadric.* □

2.7. Complex projective conics. A quadric on the projective plane is called *a projective conic*. A projective conic over $\mathbb{C}$, up to an isomorphism, coincides either with *a double line* $x_0 = 0$, which has corank 2, or with *a reducible conic*[1] $x_0^2 + x_1^2 = 0$, which has corank 1, or with the non singular conic $x_0^2 + x_1^2 + x_2^2 = 0$. The space of all conics in $\mathbb{P}_2 = \mathbb{P}(V)$ is $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$.

2.7.1. Example: standard model for non singular conic. Let U be 2-dimensional vector space. Recall that the quadratic Veronese map

$$\mathbb{P}(U^*) = \mathbb{P}_1 \xhookrightarrow{v} \mathbb{P}_2 = \mathbb{P}(S^2U^*) \tag{2-1}$$

sends a linear form ξ to its square ξ^2 (comp. with (n° 5.4.1)). If we think of $\mathbb{P}(S^2U^*)$ as the space of quadrics on $\mathbb{P}(U)$, then the Veronese embedding is a bijection between the points of $\mathbb{P}_1$ and the singular quadrics on $\mathbb{P}_1$, which are the double points. Thus, the image of (2-1) is the projective conic

$$Q_V = \{\, q \in S^2U^* \mid \det q = 0 \,\} , \tag{2-2}$$

consisting of singular quadrics on $\mathbb{P}_1$. It is called *the Veronese conic*.

Let us fix a basis (x_0, x_1) for U^*, induced basis $\{x_0^2, 2\,x_0x_1, x_1^2\}$ for S^2U^*, and write $\xi \in U^*$, $q \in S^2U^*$ as $\xi(x) = t_0x_0 + t_1x_1$, $q(x) = q_0\,x_0^2 + 2\,q_1\,x_0x_1 + q_2\,x_1^2$. Using $(t_0 : t_1)$ and $(q_0 : q_1 : q_2)$ as homogeneous coordinates on $\mathbb{P}(U^*)$ and $\mathbb{P}(S^2(U^*))$, we can describe the conic (2-3) by equation

$$q_0q_2 - q_1^2 = 0 \tag{2-3}$$

and write the Veronese embedding (2-1) as

$$(t_0 : t_1) \longmapsto (q_0 : q_1 : q_2) = (t_0^2 : t_0t_1 : t_1^2) . \tag{2-4}$$

This gives precise homogeneous quadratic parameterization for non singular conic (2-3). If $\Bbbk$ is algebraically closed, then any non singular conic $Q \subset \mathbb{P}_2$ can be identified with Q_V by an appropriate basis choice. This gives another way to produce a quadratic parameterization for a smooth plane conic besides one described in n° 1.9.1, where we used a projection of the conic onto a line from a point lying on the conic.

2.7.2. PROPOSITION. *Two distinct non singular conics have at most 4 intersection points.*

Proof. Taking appropriate coordinates, we can identify the first conic with the Veronese conic, which has quadratic parameterization $x = v(t_0, t_1)$. If the second conic is given by an equation $q(x) = 0$, then the t-parameters of the intersection points satisfy the 4-th degree equation $q(v(t)) = 0$. □

[1] i. e. a pair of crossing lines

2.7.3. COROLLARY. *Any 5 points in $\mathbb{P}_2$ lay on some conic. It is unique iff no 4 of the points are collinear. If no 3 of the points are collinear, then this conic is non singular.*

Proof. The existence of a conic follows from n° 2.6.1. Since a singular conic is either a pair of crossing lines or a double line, any quintuple of its points contains a triple of collinear points. Thus, if no 3 of 5 points are collinear, a conic is smooth and unique by the previous proposition. If the quintuple contains a triple of collinear points, then the line passing through this triple has to be a component of any conic containing the quintuple. This forces the conic to split into the union of this line and the line joining two remaining points. □

2.8. Complex projective quadrics on $\mathbb{P}_3$, up to isomorphism, are: *a double plane* $x_0^2 = 0$; *a reducible quadric* $x_0^2 + x_1^2 = 0$, which is a pair of crossing planes (or a cone with a line vertex over a pair of distinct points on an complementary line); *a simple cone* $x_0^2 + x_1^2 + x_2^2 = 0$, which is a cone with one point vertex over a non singular plane conic; and a non singular quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. The non singular quadric also has much more convenient determinantal model called *the Segre quadric* and described as follows.

Let us fix a pair of 2-dimensional vector spaces U_-, U_+ and write $W = \mathrm{Hom}(U_-, U_+)$ for the space of all linear maps $U_- \longrightarrow U_+$. Then $\mathbb{P}_3 = \mathbb{P}(W)$ consists of non zero linear maps considered up to proportionality and can be identified with the space of non zero 2×2 - matrices $\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}$ up to a scalar factor. By the definition, *the Segre quadric*

$$Q_S = \left\{ U_- \xrightarrow{F} U_+ \;\middle|\; \mathrm{rk}\, F = 1 \right\} = \left\{ A = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \;\middle|\; \det(A) = \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = 0 \right\} \tag{2-5}$$

consists of all non zero but degenerate linear maps. It coincides with the image of *the Segre embedding*

$$\mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \overset{s}{\hookrightarrow} \mathbb{P}(\mathrm{Hom}(U_-, U_+)) = \mathbb{P}_3$$

that sends $(\xi, v) \in U_-^* \times U_+$ to the rank 1 operator $\xi \otimes v \,:\, U_- \xrightarrow{u \mapsto \xi(u)\cdot v} U_+$, whose image is spanned by v and the kernel is given by the linear equation $\xi = 0$.

Indeed, any rank one operator $U_- \xrightarrow{F} U_+$ has 1-dimensional kernel, say, spanned by some $v \in U_+$. Then F has to take any $u \in U_-$ to $F(u) = \xi(u)\cdot v$, where the coefficient $\xi(u)$ is $\Bbbk$-linear in u, i. e. $\xi \in U_-^*$. Thus, $F = \xi \otimes v$ and both ξ, v are unique up to proportionality.

Exercise 2.3. Show that any $m \times n$ matrix A of rank one can be obtained as a matrix product of m-column and n-row: $A = {}^t\xi \cdot v$ for appropriate $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \Bbbk^m$, $v = (x_1, x_2, \dots, x_n) \in \Bbbk^n$, which are unique up to proportionality.

Fixing some coordinates $(\xi_0 : \xi_1)$ in U_-^* and $(t_0 : t_1)$ in U_+, we can write the operator $\xi \otimes v$ by the matrix

$$\xi \otimes v = \begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix} .$$

So, the Segre embedding gives a rational parametrization

$$\alpha_{00} = \xi_0 t_0 \;, \quad \alpha_{01} = \xi_1 t_0 \;, \quad \alpha_{10} = \xi_0 t_1 \;, \quad \alpha_{11} = \xi_1 t_1 \;. \tag{2-6}$$

for the quadric (2-5), where the parameter $\big((\xi_0 : \xi_1) \,,\, (t_0 : t_1) \big) \in \mathbb{P}_1 \times \mathbb{P}_1$. Note that $\mathbb{P}_1 \times \mathbb{P}_1$ is ruled by two families of «coordinate» lines $\xi \times \mathbb{P}_1$ and $\mathbb{P}_1 \times t$. Let us call them *the first* and *the second* ruling line families. Since the parameterization (2-6) takes lines to lines, we get

2.8.1. LEMMA. *The Segre embedding sends each coordinate line family to the ruling of Q_S by a family of pairwise skew lines. These two line families exhaust all the lines on Q_S. Any two lines from different families are intersecting and each point of Q_S is the intersection point of two lines from different families.*

Proof. A line $\xi \times \mathbb{P}_1$, where $\xi = (\xi_0 : \xi_1) \in \mathbb{P}(U_-^*)$, goes to the set of all rank 1 matrices with the ratio

$$(\text{1-st column}) : (\text{2-nd column}) = \xi_0 : \xi_1 \;.$$

They form a line in $\mathbb{P}_3$ given by two linear equations $a_{00} : a_{01} = a_{10} : a_{11} = \xi_0 : \xi_1$. Analogously, $s(\mathbb{P}_1 \times t)$, where $t = (t_0 : t_1) \in \mathbb{P}(U_+)$, goes to the line given in $\mathbb{P}_3$ by $a_{00} : a_{10} = a_{01} : a_{11} = t_0 : t_1$ and formed by all rank 1 matrices with the ratio

$$(\text{1-st row}) : (\text{2-nd row}) = t_0 : t_1 \;.$$

Since the Segre embedding is bijective, each line family consists of pairwise skew lines, any two lines from the different families are intersecting, and for any $x \in Q_s$ there is a pair of from the different families that are intersecting at x. This forces $Q_s \cap T_x Q_s$ to be a split conic and implies that there are no other lines on Q_s. □

2.8.2. COROLLARY. *Any 3 lines on $\mathbb{P}_3$ lie on some quadric. If the lines are mutually skew, then this quadric is unique, non singular, and is ruled by all lines in $\mathbb{P}_3$ intersecting all 3 given lines.*

Proof. The space of quadrics in $\mathbb{P}_3$ has dimension $3 \cdot 6 : 2 = 9$. Thus, any 9 points in $\mathbb{P}_3$ lay on some quadric. If we pick up a triple of distinct points on each line and draw a quadric through these 9 points, then this quadric will contain all 3 lines (comp. with n° 2.2). Since a singular quadric does not contain a triple of mutually skew lines, any quadric passing through 3 pairwise skew lines is non singular and is ruled by two families of lines. Clearly, the triple of given lines lies in the same family. Then the second family can be described geometrically as the set of lines in $\mathbb{P}_3$ intersecting all 3 given lines. Thus the quadric is unique. □

Exercise 2.4. How many lines intersect 4 given pairwise skew lines in $\mathbb{P}_3$?

Exercise 2.5*. How will the answer be changed, if we replace a) $\mathbb{P}_3$ by $\mathbb{A}^3$ b) $\mathbb{C}$ by $\mathbb{R}$? Find all possible solutions and indicate those that are stable under small perturbations of the initial configuration of 4 lines.

2.9. Linear subspaces lying on a non singular quadric. The line rulings from n° 2.8.1 have higher dimensional versions as well. Let $Q_n \subset \mathbb{P}_n = \mathbb{P}(V)$ be non singular quadric and $L = \mathbb{P}(W)$ be a projective subspace lying on Q_n.

2.9.1. THEOREM. $\dim L \leqslant \left[\dfrac{n-1}{2}\right]$, *where $[\,*\,]$ means the integer part.*

Proof. Let Q_n be given by a quadratic form q with the polarization $\widetilde{q}$. Then

$$L \subset Q_n \iff \widetilde{q}(w_1, w_2) = 0 \;\forall\, w_1, w_2 \in W \iff \widehat{q}(W) \subset \operatorname{Ann}(W) = \{\, \xi \in V^* \mid \xi(w) = 0 \quad \forall\, w \in W \,\}\,,$$

where $\widehat{q} : v \longmapsto \widetilde{q}(v, *)$ is the correlation associated with Q_n. Since Q_n smooth, this correlation is injective. Thus, $\dim W \leqslant \dim \operatorname{Ann} W = \dim V - \dim W$ and $\dim L = \dim W - 1 \leqslant (\dim V)/2 - 1 = [(n-1)/2]$. □

2.9.2. LEMMA. *$\operatorname{cork}(H \cap Q_n) \leqslant 1$ for any codimension 1 hyperplane $H \subset \mathbb{P}_n$.*

Proof. If $H = \mathbb{P}(W)$, then $\dim \ker (q|_W) \leqslant \dim\big(W \cap \widehat{q}^{-1}(\operatorname{Ann} W)\big) \leqslant \dim \widehat{q}^{-1}(\operatorname{Ann} W) = \dim \operatorname{Ann} W = 1$. □

2.9.3. LEMMA. *For any $x \in Q_n$ the intersection $Q_n \cap T_x Q_n$ is a simple cone with the vertex x over a non singular quadric Q_{n-2} in an $(n-2)$-dimensional projective subspace in $T_x Q_n \setminus \{x\}$.*

Proof. Since $T_x Q = \mathbb{P}(\ker \widetilde{q}(x, *))$ and $\widetilde{q}(x, x) = q(x) = 0$, the restriction of q onto $T_x Q$ has at least 1-dimensional kernel presented by x itself. By the previous lemma this kernel is spaned by x. □

2.9.4. THEOREM. *Let $d_n = [(n-1)/2]$ be the upper bound from n° 2.9.1 and $x \in Q_n$ be an arbitrary point. Then d_n-dimensional subspaces $L \subset Q_n$ passing through x stay in 1–1 correspondence with $(d_n - 1)$-dimensional subspaces lying on Q_{n-2}.*

Proof. Fix some $(n-1)$-dimensional projective subspace $H \subset T_x Q \setminus \{x\}$ and present $Q_n \cap T_x Q_n$ as a simple cone ruled by lines passing through x and some $Q_{n-2} \subset H$. Since any $L \subset Q_n$ which pass through x is contained inside $Q_n \cap T_x Q_n$, it has to be the linear span of x and some $(d_n - 1)$-dimensional subspace $L' \subset Q_{n-2}$. □

For example, there are only 0-dimensional subspaces on Q_1 and Q_2. Next two quadrics, Q_3 and Q_4, do not contain planes. But any point $x \in Q_3$ lies on two lines passing through x and 2 points of $Q_1 \subset T_x Q_3 \setminus \{x\}$ and any point of Q_4 belongs to 1-dimensional family of lines parameterized by the points of a non singular conic $Q_2 \subset T_x Q_4 \setminus \{x\}$. Further, non singular quadric $Q_5 \subset \mathbb{P}_5$ does not contain 3-dimensional subspaces but for any point $x \in Q_5$ there are two 1-dimensional families of planes passing through x. Each family is parameterized by the corresponding family of lines on $Q_3 \subset T_x Q_5 \setminus \{x\}$, i. e. by $\mathbb{P}_1$ actually.

§3. Working examples: conics, pencils of lines, and plane drawings.

During this section we continue to assume that $\operatorname{char}\Bbbk \neq 2$.

3.1. Projective duality. For any $0 \leqslant m \leqslant (n-1)$ there is a canonical bijection between the m-dimensional projective subspaces in $\mathbb{P}_n = \mathbb{P}(V)$ and the $(n-1-m)$-dimensional ones in $\mathbb{P}_n^\times \overset{\text{def}}{=} \mathbb{P}(V^*)$. It sends a subspace $L = \mathbb{P}(U)$ to the subspace $L^\times \overset{\text{def}}{=} \mathbb{P}(\operatorname{Ann}(U))$, where

$$\operatorname{Ann}(U) \overset{\text{def}}{=} \{\,\xi \in V^* \mid \xi(u) = 0 \quad \forall\, u \in U\,\}$$

is *an annihilator* of U. Note that $L^{\times\times} = L$, since $\operatorname{Ann}\operatorname{Ann} U = \{\, v \in V \mid \xi(v) = 0 \quad \forall \xi \in \operatorname{Ann} U\,\} = U$ under the natural identification $V^{**} \simeq V$. The correspondence $L \leftrightarrow L^\times$ is called *a projective duality*. It inverts inclusions[1] and linear incidences[2]. The projective duality translates the geometry on $\mathbb{P}_n$ to the one on $\mathbb{P}_n^\times$ and back. For example, in $\mathbb{P}_2$-case we have the following dictionary:

a line $\ell \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $\ell^\times \in \mathbb{P}_2^\times$
the points p of the above line ℓ	$\longleftrightarrow$	the lines $p^\times$ passing through the above point $\ell^\times$
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point for two lines $p_1^\times, p_2^\times \subset \mathbb{P}_2^\times$
the points p of some conic Q	$\longleftrightarrow$	the tangent lines $p^\times$ of some conic $Q^\times$
the tangent lines ℓ to Q	$\longleftrightarrow$	the points $\ell^\times$ of $Q^\times$

Exercise 3.1. Explain the last two items by proving that the tangent spaces of a non singular quadric $Q \subset \mathbb{P}_n$ correspond to the points of some non singular quadric $Q^\times \subset \mathbb{P}_n^\times$. Show also that Q and $Q^\times$ have inverse Gram matrices in dual bases of V and V^*.

HINT. If $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in V^*$ and $x = (x_0, x_1, \dots, x_n) \in V$ in dual coordinate systems, then $\xi(x) = \xi \cdot {}^t x$. Let $Q \subset \mathbb{P}(V)$, $Q^\times \subset \mathbb{P}(V^*)$ have inverse Gram matrices A, A^{-1}. Since $T_x Q = \mathbb{P}(\operatorname{Ann}\xi) \iff \xi = x \cdot A \iff x = \xi \cdot A^{-1}$, we have $x \in Q \iff x \cdot A \cdot {}^t x = 0 \iff (\xi \cdot A^{-1}) \cdot A \cdot {}^t(\xi \cdot A^{-1}) = 0 \iff \xi \cdot A^{-1}\,{}^t\xi = 0 \iff \xi \in Q^\times$.

3.1.1. COROLLARY. *Any 5 lines on $\mathbb{P}_2$ without triple intersections are tangent to unique smooth conic.*

Proof. This assertion is projectively dual to n° 2.7.3. Namely, let $\ell_i \in \mathbb{P}_2$ be 5 given lines. There exists a unique conic $Q^\times$ passing through 5 points $\ell_i^\times \in \mathbb{P}_2^\times$. Then $\ell_i = \ell_i^{\times\times}$ are tangent to the dual conic $Q^{\times\times} = Q \subset \mathbb{P}_2$. $\square$

3.2. Projective linear isomorphisms $\mathbb{P}_1 \longrightarrow \mathbb{P}_1$ via conics. Consider a non singular conic Q and a line ℓ and write $Q \xrightarrow{\pi_\ell^p} \ell$ for the bijective map given by projection from a point $p \in Q$ extended into p by sending $p \longmapsto \ell \cap T_p Q$.

Exercise 3.2. Show that this bijection is given by some rational algebraic functions, which express coordinates of the corresponding points through each other.

HINT. To express the coordinates of $t = \pi_\ell^p(x)$ through x, you take special coordinates where ℓ is given by $x_0 = 0$ and $p = (1:0:0)$; then $t = (0 : x_1 : x_2)$. In order to get the inverse expression, only two skills are quite enough: linear equations solving and finding the second root for a quadratic equation with the first root known. The both procedures have a rational output.

So, if ℓ_1, ℓ_2 are two lines and p_1, p_2 are two distinct points on some non singular conic Q, then the composition $\gamma_Q^{p_2 p_1} \overset{\text{def}}{=} \pi_{\ell_2}^{p_2} \circ \left(\pi_{\ell_1}^{p_1}\right)^{-1}$ gives a projective linear isomorphism $\ell_1 \xrightarrow{\sim} \ell_2$ (see fig. fig 3◇1). In fact, any projective linear isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ can be presented[3] as $\gamma_Q^{p_2 p_1}$ for some Q and $p_1, p_2 \in Q$. Indeed, if γ sends, say $a_1, b_1, c_1 \in \ell_1$ to $a_2, b_2, c_2 \in \ell_2$, we pick *any* p_1, p_2 such that no 3 out of 5 points p_1, p_2, $(a_1 p_1) \cap (a_2 p_2)$, $(b_1 p_1) \cap (b_2 p_2)$, $(c_1 p_1) \cap (c_2 p_2)$ are collinear (see fig 3◇2) and draw Q through these 5 points. Then $\gamma = \gamma_Q^{p_2 p_1}$, because both have the same action on 3 points a_1, b_1, c_1.

[1] i. e. $L_1 \subset L_2 \iff L_1^\times \supset L_2^\times$

[2] i. e. $L_1, L_2, \dots, L_r$ are contained in some m-dimensional subspace L iff $L_1^\times, L_2^\times, \dots, L_r^\times$ contain some $(n-m-1)$-dimensional subspace $L^\times$; for example: 3 points are collinear iff their dual 3 hyperplanes have common subspace of codimension 2

[3] in infinitely many different ways

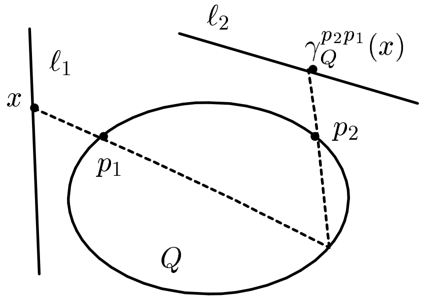


Fig. 3◇1. Composing projections.

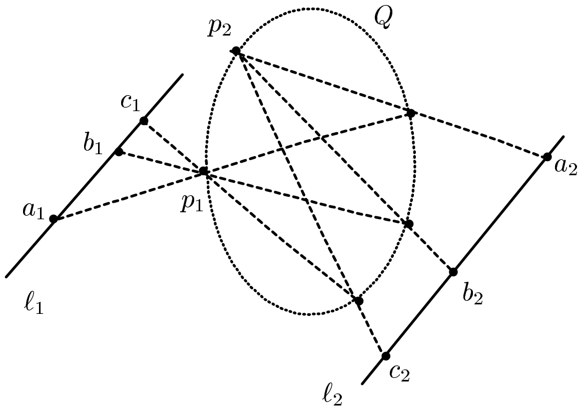


Fig. 3◇2. How to find p_1, p_2.

3.3. Drawing a conic by the ruler. Let 5 distinct points p_1, p_2, a, b, c lay on a non singular conic Q. Denote the line $(a\,c)$ by ℓ_1, the line $(b\,c)$ by ℓ_2, and the intersection point $(a\,p_2) \cap (b\,p_1)$ by O (see fig 3◇3). Then the projective linear isomorphism $\ell_1 \xrightarrow{\gamma_Q^{p_2p_1}} \ell_2$ coincides with the simple projective linear isomorphism $\ell_1 \xrightarrow{\gamma_O} \ell_2$, which takes $x \in \ell_1$ to $\gamma_O(x) = (x\,O) \cap \ell_2$ (indeed, the both send $c \longmapsto c$, $a \longmapsto d$, $e \longmapsto b$, where $d = (a\,p_2) \cap \ell_2$ and $e = (b\,p_1) \cap \ell_1$ — see fig 3◇3).

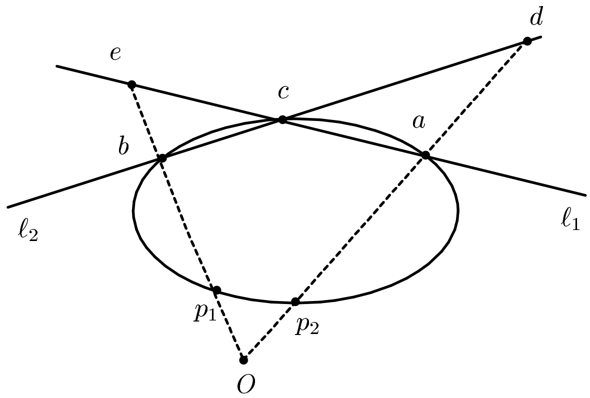


Fig. 3◇3. Remarkable coincidence.

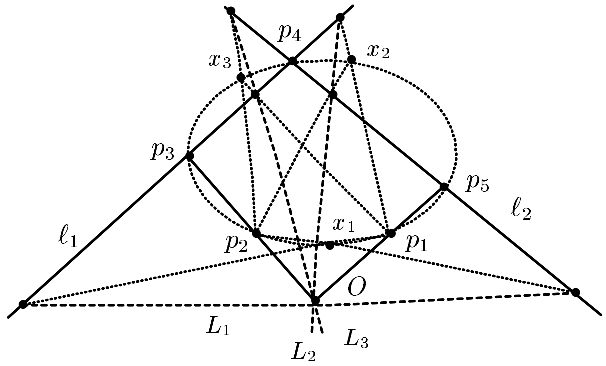


Fig. 3◇4. Tracing a conic.

This simple remark allows us to trace, using only the ruler, a dense point set on the conic passing through 5 given points $p_1, p_2, \ldots, p_5$ (see fig 3◇4). Namely, let $\ell_1 = (p_3p_4)$, $\ell_2 = (p_4p_5)$, $O = (p_1p_6) \cap (p_2p_3)$. Then any line $L \ni O$ gives two intersection points $\ell_1 \cap L$ and $\ell_2 \cap L$. These points are sent to each other by the projective linear isomorphism $\gamma_O = \gamma_Q^{p_2p_1}$. So, if we draw lines through p_1 and $\ell_1 \cap L$, and trough p_2 and $\ell_2 \cap L$, then the intersection point x of these two lines has to lay on Q. On the fig 3◇4 the points x_1, x_2, x_3 are constructed by this way starting from the lines L_1, L_2, L_3 passing through O.

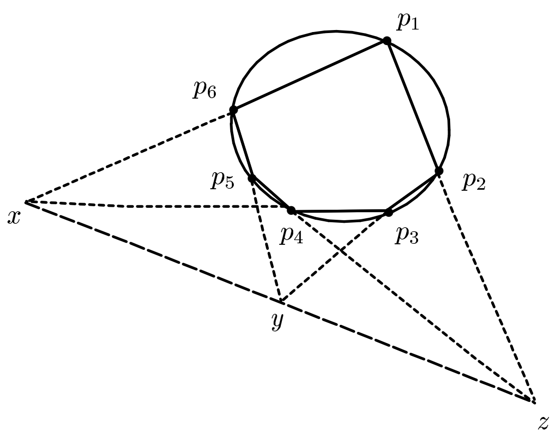


Fig. 3◇5. Inscribed hexagon.

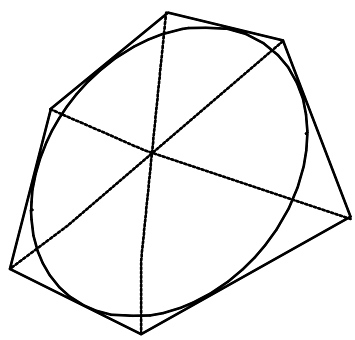


Fig. 3◇6. Circumscribed hexagon.

3.3.1. PROPOSITION (PASCAL'S THEOREM). *A hexagon $p_1, p_2, \dots, p_6$ is inscribed into a non singular conic iff the points*[1] $(p_1p_2) \cap (p_4p_5)$, $(p_2p_3) \cap (p_5p_6)$, $(p_3p_4) \cap (p_6p_1)$ *are collinear (see fig 3◇5).*

Proof. Draw the conic Q through 5 of p_i except for p_4 and put $\ell_1 = (p_1p_6)$, $\ell_2 = (p_1p_2)$, $y = (p_5p_6) \cap (p_2p_3)$, $x = (p_3p_4) \cap \ell_1$, $z = (xy) \cap \ell_2$. Then $z \in \ell_2$ is the image of $x \in \ell_1$ under the projective linear isomorphism$\gamma_y = \gamma^{p_5p_3} : \ell_1 \xrightarrow{\sim} \ell_2$ like before. In particular, the intersection point $(p_3p_4) \cap (p_5z)$ lays on Q. Hence, $p_4 \in Q$ iff $p_4 = (p_3p_4) \cap (p_5z)$. □

3.3.2. COROLLARY (BRIANCHON'S THEOREM). *A hexagon $p_1, p_2, \dots, p_6$ is circumscribed around a non singular conic iff its main diagonals (p_1p_4), (p_2p_5), (p_3p_6) are intersecting at one point (see fig. fig 3◇6).*

Proof. This is just the projectively dual version of the Pascal theorem. □

3.4. Linear isomorphisms of pencils. A family of geometrical figures is referred as *a pencil*, if it is naturally parameterized by the projective line. For example, all lines passing through a given point $p \in \mathbb{P}_2$ form a pencil, because their equations run trough the line $p^\times \in \mathbb{P}_2^\times$ by projective duality. More generally, there is a pencil of hyperplanes $H \subset \mathbb{P}_n$ passing through a given subspace $L \subset \mathbb{P}_n$ of codimension 2. Such a pencil is denoted by $|h - L|$ (read: «all hyperplanes containing L») or by $L^\times \in \mathbb{P}_n^\times$. Given two such pencils, say $L_1^\times$, $L_2^\times$ and 3 points $a, b, c \in \mathbb{P}_n \setminus (L_1 \cup L_2)$ such that 3 hyperplanes from $L_i^\times$ passing through them are distinct in the both pencils, then these 3 points define a projective linear isomorphism $L_1^\times \xrightarrow{\gamma_{abc}} L_2^\times$ that sends 3 hyperplanes of the first pencil passing through a, b, c to the corresponding ones from the second pencil.

3.4.1. Example: linear identification of two pencils $p_1^\times$ and $p_2^\times$ on $\mathbb{P}_2$ is given by any 3 points a, b, c such that any 2 of them are not collinear with p_1 or p_2. It sends $(p_1a) \longmapsto (p_2a)$, $(p_1b) \longmapsto (p_2b)$, $(p_1c) \longmapsto (p_2c)$. Let Q be the (unique!) conic passing trough 5 points p_1, p_2, a, b, c. There are two different cases (see fig 3◇7–fig 3◇8).

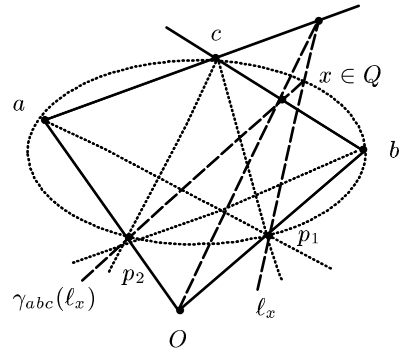


Fig. 3◇7. Elliptic isomorphism of pencils.

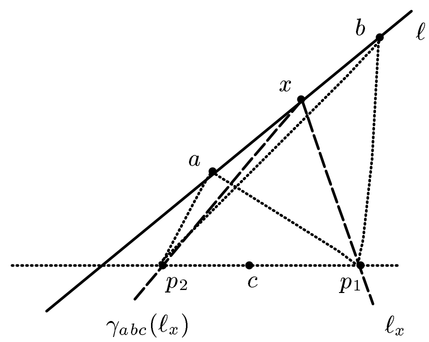


Fig. 3◇8. Parabolic isomorphism of pencils.

(A) Elliptic case: Q is non singular, i. e. all 5 points are linearly general. In this case *the incidence graph*[2] of γ_{abc} coincides with Q, because the points of Q give the projective linear isomorphism $p_1^\times \longrightarrow p_2^\times$ that has the same action on a, b, c. Moreover, the above discussions let us draw the line $\gamma_{abc}(\ell_x)$ for a given $(\ell_x \in p_1^\times)$ by the ruler as follows (see fig 3◇7). First mark the point $O = (p_1b) \cap (p_2c)$; then find the intersection point $\ell_x \cap (a\,c)$, join it with O by a line, and mark the point where this line crosses $(b\,c)$; then the line $\gamma_{abc}(\ell_x)$ goes through this marked point.

(B) Parabolic case: Q is reducible, i. e. splits in two lines: (p_1p_2) and, say, $\ell = (a\,b)$. This happens when $c \in (p_1p_2)$ (recall that no 2 points from a, b, c are collinear with any of p_i). In this case the incidence graph for γ_{abc} coincides with the line ℓ (see fig 3◇8).

Dualizing these examples, we get geometrical classification of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$ between two given lines on $\mathbb{P}_2$.

3.4.2. COROLLARY. *There are exactly two types of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$. Elliptic isomorphisms γ_Q correspond bijectively to the non singular conics Q touching both ℓ_1 and ℓ_2. Such γ_Q sends $x \longmapsto y$ iff the line $(x\,y)$ is tangent to Q (see fig 3◇9). Parabolic isomorphisms γ_L are parameterized by the points $L \in \mathbb{P}_2 \setminus (\ell_1 \cup \ell_2)$. Such γ_L sends $x \longmapsto y$ iff the line $(x\,y)$ pass through L (see fig 3◇10).* □

[1] i. e. 3 intersection points of the line pairs passing through the opposite sides of the hexagon

[2] i. e. a curve traced by the intersection points $\ell \cap \gamma_{abc}(\ell)$ while ℓ runs through $p_1^\times$

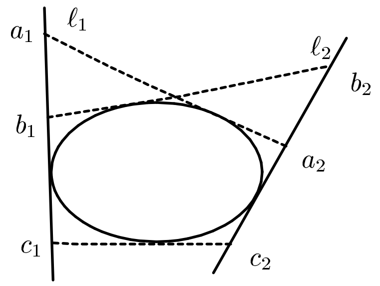


Fig. 3◇9. Elliptic isomorphism of lines.

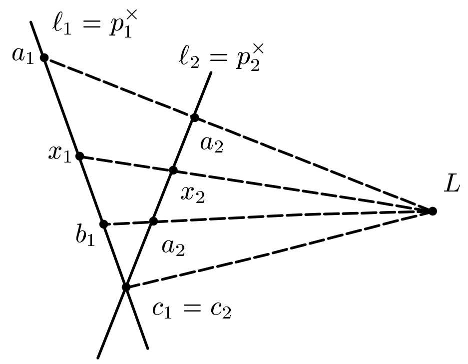


Fig. 3◇10. Parabolic isomorphism of lines.

3.5. Towards Poncelet's porism. Given two non singular conics Q, Q', we can try to draw an n-gone simultaneously inscribed in Q' and circumscribed about Q: starting from some point $p_1 \in Q'$ draw a tangent line from p_1 to Q until it meets Q' in p_2, then draw a tangency from p_2 e. t. c. Poncelet's theorem says that if this procedure comes back to $p_n = p_1$ after n steps, then the same holds for any choice of the starting point p_1 maybe except for some finite set. The next two corollaries explain Poncelet porism for triangles (i. e. for $n = 3$)

3.5.1. COROLLARY. *Two triangles ABC and $A'B'C'$ are both inscribed into the same conic Q iff they are both circumscribed around the same conic Q'.*

Proof. We check only «⇒» implication, then the opposite implication comes by projective duality. Consider two lines $\ell = (AB)$, $\ell' = (A'B')$ and elliptic projective linear isomorphism $\ell \xrightarrow{\gamma} \ell'$ composed as the projection of ℓ onto Q from B' followed by the projection of Q onto ℓ' from B (see fig 3◇11). Since it takes $A \mapsto L'$, $C \mapsto K'$, $K \mapsto C'$, $L \mapsto A'$, all the sides of the both triangles should touch the conic associated with γ via n° 3.4.2. □

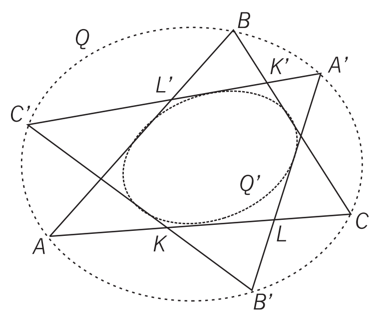


Fig. 3◇11. Inscribed-circumscribed triangles.

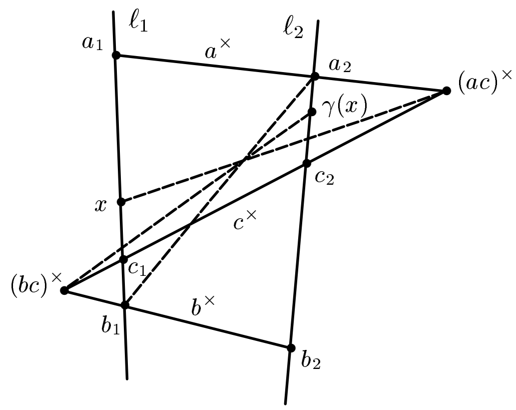


Fig. 3◇12. Finding $\gamma_Q(x)$.

3.5.2. COROLLARY. *Given two conics Q, Q' such that there exists a triangle ABC inscribed into Q and circumscribed around Q', then any point $A' \in Q$ is a vertex of a triangle $A'B'C'$ inscribed into Q and circumscribed around Q'.*

Proof. Take any $A' \in Q$ and pick $B', C' \in Q$ such that the lines $(A'B')$, $(A'C')$ are tangent to Q' (see fig 3◇11 again). By the previous corollary, both ABC and $A'B'C'$ are circumscribed around some conic, which must coincide with Q', because there exist a unique conic touching 5 lines (AB), (BC), (CA), $(A'B')$, $(A'C')$. □

Exercise 3.3. Make n° 3.1.1 more precise by finding necessary and sufficient condition on 5 lines in $\mathbb{P}_2$ for existence of a unique non singular conic touching all of them.

HINT. This is projectively dual to n° 2.7.3

3.6. Making a projective isomorphism by the ruler. If an isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ is given by its action on some 3 points, say: $a_1 \longmapsto a_2$, $b_1 \longmapsto b_2$, $c_1 \longmapsto c_2$, then we can find the image $\gamma(x)$ of any

$x \in \ell_1$ by the ruler. In parabolic case this is trivial (see fig 3◇10). In elliptic case the drawing algorithm is projectively dual to the one discussed in n° 3.4.1 (A). Namely, draw the line $O^\times = (b_1 a_2)$; then pass the line through x and $(a_1 a_2) \cap (c_1 c_2)$ and mark its intersection point with $O^\times$; now $\gamma(x)$ is the intersection of ℓ_2 with the line passing through the last marked point and $(b_1 b_2) \cap (c_1 c_2)$ (compare fig 3◇7 and fig 3◇12).

Exercise 3.4. Let $Q \subset \mathbb{P}_2$ be non singular conic considered together with some rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ a map $Q \xrightarrow{\gamma} Q$ given by prescription: $x \overset{\gamma}{\longmapsto} y \iff \pi_\ell^{p_1} x = \pi_\ell^{p_2} y$ is induced by some linear automorphism of $\mathbb{P}_1$ (i. e. by some linear fractional reparameterization). Find the images of p_1, p_2 and the fixed points of the above map. Show that any bijection $Q \xrightarrow{\sim} Q$ induced by a linear automorphism of $\mathbb{P}_1$ can be (not uniquely) realized geometrically by two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ in the way described above. Is it possible, using only the ruler, to find (some) p_1, p_2, ℓ for a bijection $Q \xrightarrow{\sim} Q$ given by its action on 3 points $a, b, c, \in Q$?

HINT. Try $p_2 = a$.

Exercise 3.5*. Given a non singular conic Q and three points A, B, C, draw (using only the ruler) a triangle inscribed in Q with sides passing through A, B, C. How many solutions may have this problem?

HINT. Start «naive» drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in ex. 3.4?

Exercise 3.6*. Formulate and solve projectively dual problem to the previous one.

§4. Tensor Guide.

4.1. Multilinear maps. Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over an arbitrary field $\Bbbk$. A map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ is called *multilinear*, if in each argument

$$\varphi(\dots, \lambda v' + \mu v'', \dots) = \lambda\, \varphi(\dots, v', \dots) + \mu\, \varphi(\dots, v'', \dots)$$

when all the other remain to be fixed. The multilinear maps $V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ form a vector space of dimension $m \cdot \prod d_\nu$. Namely, if we fix a basis $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\}$ for each V_i and a basis $\{e_1, e_2, \dots, e_m\}$ for W, then any multilinear map φ is uniquely defined by its values at all combinations of the basic vectors:

$$\varphi\big(\, e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}\big) = \sum_\nu a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot e^\nu \in W$$

As soon as $m \cdot \prod d_\nu$ numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \in \Bbbk$ are given, the map φ is well defined by the multilinearity. It sends a collection of vectors $(v_1, v_2, \dots, v_n)$, where $v_i = \sum\limits_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)}\, e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{j=1}^{m} \Big(\sum_{\alpha_1, \alpha_2, \dots, \alpha_n} a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \cdots \cdot x_{\alpha_n}^{(n)} \Big) \cdot e^\nu \in W$$

the numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ can be considered as elements of some «$(n+1)$-dimensional format matrix of size $m \times d_1 \times d_2 \times \cdots \times d_n$», if you can imagine such a thing[1].

Exercise 4.1. Check that a collection $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ doesn't contain zero vector iff there exists a multilinear map φ (to somewhere) such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$.

Exercise 4.2. Check that a multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} U$ composed with *a linear operator* $U \xrightarrow{F} W$ is a multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{F \circ \varphi} W$ as well.

4.2. Tensor product of vector spaces. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ be a fixed multilinear map. Then for any vector space W we have the composition operator

$$\begin{pmatrix} \text{the space } \operatorname{Hom}(U, W) \text{ of all} \\ \text{linear operators } U \xrightarrow{F} W \end{pmatrix} \xrightarrow{\quad F \longmapsto F \circ \tau \quad} \begin{pmatrix} \text{the space of all multilinear maps} \\ V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W \end{pmatrix} \tag{4-1}$$

A multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ is called *universal* if the composition operator (4-1) is an isomorphism for any vector space W. In other words, the multilinear map τ is universal, if for any W and any multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ there exist a unique linear operator $U \xrightarrow{F} W$ such that $\varphi = F \circ \tau$, i. e. the commutative diagram

$$\begin{tikzcd}
 & U \arrow[dd, dashed, "F"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[ur, "\tau"] \arrow[dr, "\varphi"'] & \\
 & W
\end{tikzcd}$$

can be always closed by a unique linear dotted row.

4.2.1. CLAIM. *Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_1} U_1$ и $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_2} U_2$ be two universal multilinear maps. Then there exists a unique linear isomorphism $U_1 \xrightarrow{\iota} U_2$ such that $\tau_2 = \iota\tau_1$.*

[1]a usual $d \times m$ - matrix, which presents a *linear* map $V \longrightarrow W$, has just 2-dimensional format

Proof. Since both U_1, U_2 are universal, there are unique linear operators $U_1 \xrightarrow{F_{21}} U_2$ and $U_2 \xrightarrow{F_{12}} U_1$ mounted in the diagrams

$$\begin{tikzcd}
U_1 \arrow[rr, equal, "\mathrm{Id}_{U_1}"] & & U_1 \arrow[dd, dashed, "F_{21}"] \\
& V_1 \times V_2 \times \cdots \times V_n \arrow[ul, "\tau_1"'] \arrow[ur, "\tau_1"] \arrow[dl, "\tau_2"] \arrow[dr, "\tau_2"'] & \\
U_2 \arrow[uu, dashed, "F_{12}"] \arrow[rr, equal, "\mathrm{Id}_{U_2}"'] & & U_2
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
& & U_2 \arrow[dl, dotted, "F_{12}"] \arrow[dd, equal, "\mathrm{Id}_{U_2}"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[r, "\tau_1"] \arrow[urr, "\tau_2"] \arrow[drr, "\tau_2"'] & U_1 \arrow[dr, dotted, "F_{21}"] & \\
& & U_2
\end{tikzcd}$$

So, the composition $F_{21}F_{12} = \mathrm{Id}_{U_2}$, because of the uniqueness property in the universality of U_2. Similarly, $F_{12}F_{21} = \mathrm{Id}_{U_1}$. $\square$

4.2.2. CLAIM. *Let $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\} \in V_i$ be a basis (for $1 \leqslant i \leqslant n$). Denote by $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ a $(\prod d_i)$ - dimensional vector space whose basic vectors are the symbols*

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\,, \quad 1 \leqslant \alpha_i \leqslant d_i \tag{4-2}$$

(all possible formal «tensor products» of basic vectors $e_\nu^{(\mu)}$). Then the multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

which sends a basis vector collection $(e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n$ to the corresponding basis vector (4-2) *is universal.*

Proof. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ be a multilinear map and $V_1 \otimes V_2 \otimes \cdots \otimes V_n \xrightarrow{F} W$ be a linear operator. Comparing the values at the basic vectors, we see that

$$\varphi = F \circ \tau \iff F(\, e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\,) = \varphi(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)})\,.$$

$\square$

4.3. The Segre embedding. The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor product* of $V_1, V_2, \dots, V_n$. The universal multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor multiplication*. For a collection of vectors $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ the image $\tau(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and called *a tensor product* of these vectors. All such products are called *decomposable tensors*. Of course, not all the vectors of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ are decomposable and $\operatorname{im}\tau$ is not a vector subspace in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, because τ is *multi*linear but not linear. However, the linear span of decomposable tensors exhausts the whole of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Geometrically, the tensor multiplication gives a map

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \xhookrightarrow{s} \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$$

called *a Segre embedding*. If $d_i = \dim V_i = m_i + 1$, then the Segre embedding is a bijection between $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \times \cdots \times \mathbb{P}_{m_n}$ and *a Segre variety* formed by all decomposable tensors considered up to proportionality. This variety lives in $\mathbb{P}_N$ with $N = -1 + \prod(1 + m_i)$ and has dimension $(\sum m_i)$ but does not lie in a hyperplane. It is ruled by n families of linear subspaces.

4.3.1. Example: the Segre embedding $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \hookrightarrow \mathbb{P}_{m_1+m_2+m_1m_2}$ sends $x = (x_0 : x_1 : \ldots : x_{m_1}) \in \mathbb{P}_{m_1}$ and $y = (y_0 : y_1 : \ldots : y_{m_2}) \in \mathbb{P}_{m_2}$ to the point $s(x,y) \in \mathbb{P}_{m_1+m_2+m_1m_2}$ whose $(1+m_1)(1+m_2)$ homogeneous coordinates are all possible products x_iy_j with $0 \leqslant i \leqslant m_1$ and $0 \leqslant j \leqslant m_2$. To visualize this thing, take $\mathbb{P}_{m_1} = \mathbb{P}(V^*)$, $\mathbb{P}_{m_2} = \mathbb{P}(W)$, and $\mathbb{P}_{m_1+m_2+m_1m_2} = \mathbb{P}(\operatorname{Hom}(V,W))$, where $\operatorname{Hom}(V,W)$ is the space of all linear maps. Then the Segre map sends a pair $(\xi, w) \in V^* \times W$ to the linear map $\xi \otimes w$, which acts by the rule $v \longmapsto \xi(v) \cdot w$.

Exercise 4.3. Check that a map $V^* \times W \longrightarrow \operatorname{Hom}(V,W)$ which sends (ξ, w) to the operator $v \longmapsto \xi(v) \cdot w$ is *the universal bilinear map* (so, there is a canonical isomorphism $V^* \otimes W \simeq \operatorname{Hom}(V,W)$)

Exercise 4.4. Check that for $\xi = (x_0, x_1, \dots, x_{m_1}) \in V^*$ and $w = (y_0, y_1, \dots, y_{m_2}) \in W$ operator $\xi \otimes w$ has the matrix $a_{ij} = x_j y_i$.

Since any operator $\xi \otimes w$ has 1-dimensional image, the corresponding matrix has rank 1. On the other side, any rank 1 matrix has proportional columns. Hence, the corresponding operator has 1-dimensional image, say spaned by $w \in W$, and takes $v \longmapsto \xi(v)\, w$, where the coefficient $\xi(v) \in \Bbbk$ depends on v linearly. So, the image of the Segre embedding consists of all rank 1 operators up to proportionality. In particular, it can be defined by quadratic equations $\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{ij} a_{\ell k} - a_{ik} a_{\ell j} = 0$ saying that all 2×2 - minors for the matrix $(a_{\mu\nu})$ vanish.

4.4. Tensor algebra of a vector space. If $V_1 = V_2 = \dots = V_n = V$, then $V^{\otimes n} \stackrel{\text{def}}{=} \underbrace{V \otimes V \otimes \dots \otimes V}_{n}$ is called *an n-th tensor power* of V. All tensor powers are combined in the infinite dimensional non commutative graded algebra $\mathsf{T}^{\bullet} V = \bigoplus_{n \geqslant 0} V^{\otimes n}$, where $V^{\otimes 0} \stackrel{\text{def}}{=} \Bbbk$.

Exercise 4.5. Using the universality, show that there are canonical isomorphisms

$$\left(V^{\otimes n_1} \otimes V^{\otimes n_2}\right) \otimes V^{\otimes n_3} \simeq V^{\otimes n_1} \otimes \left(V^{\otimes n_2} \otimes V^{\otimes n_3}\right) \simeq V^{\otimes (n_1 + n_2 + n_3)}$$

which make the vector's tensoring to be well defined associative multiplication on $\mathsf{T}^{\bullet} V$.

Algebraically, $\mathsf{T}^{\bullet} V$ is what is called «a free associative $\Bbbk$-algebra generated[1] by V». Practically, this means that if we fix a basis $\{e_1, e_2, \dots, e_d\} \subset V$, then $\mathsf{T}^{\bullet} V$ turns into the space of the formal finite linear combinations of words consisting of the letters e_i separated by $\otimes$. These words are multiplied by writing after one other consequently and the multiplication is extended onto linear combinations of words by the usual distributivity rules.

4.5. Duality. The spaces $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$ and $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{n}$ are canonically dual to each other. The pairing between $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ is given by *a full contraction*

$$\langle\, v \,,\, \xi \,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \xi_i(v_i)\,. \tag{4-3}$$

Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{\xi_1, \xi_2, \dots, \xi_n\} \subset V^*$ be some dual bases. Then the basic words $\{\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \,\}$ and $\{\, \xi_{j_1} \otimes \xi_{j_2} \otimes \cdots \otimes \xi_{j_s} \,\}$ form dual bases for $\mathsf{T}^{\bullet} V$ and $\mathsf{T}^{\bullet} V^*$ with respect to the full contraction. So, $V^{\otimes n *} \simeq V^{* \otimes n}$. On the other side, the space $\left(V^{\otimes n}\right)^*$ is naturally identified with the space of all multilinear forms $\underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow \Bbbk$, because $V^{\otimes n}$ is universal. So, there exists a canonical isomorphism between $V^{*\otimes n} = \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{n}$ and the space of multilinear forms in n arguments from V. It sends a tensor $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ to the form $(v_1, v_2, \dots, v_n) \longmapsto \prod_{i=1}^{n} \xi_i(v_i)$.

4.6. Partial contractions. Let $\{1, 2, \dots, p\} \xleftarrow{I} \{1, 2, \dots, m\} \xrightarrow{J} \{1, 2, \dots, q\}$ be two injective (not necessary monotonous) maps. We write i_ν and j_ν for $I(\nu)$ and $J(\nu)$ respectively and consider $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$ as two ordered (but not necessary monotonous) index collections of the same cardinality. A linear operator

$$\underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{p} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q} \xrightarrow{c^I_J} \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{p-m} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{q-m}$$

which sends $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q$ to $\prod_{\nu=1}^{m} \xi_{i_\nu}(v_{j_\nu}) \cdot \bigotimes_{i \notin \operatorname{im}(I)} \xi_i \otimes \bigotimes_{j \notin \operatorname{im}(J)} v_j$ is called *a partial contraction* in the indexes I and J.

[1] If you like it, make the following formal exercise: deduce from the universality that for any associative $\Bbbk$-algebra A and vector space map $V \xrightarrow{f} A$ there exists a unique algebra homomorphism $\mathsf{T}^{\bullet} V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

4.6.1. Example: the contraction between a vector and a multilinear form. Consider a multilinear form

$$\varphi(v_1, v_2, \dots, v_n)$$

as a tensor from $V^{*\otimes n}$ and contact it in the first index with a vector $v \in V$. The result belongs to $V^{*\otimes(n-1)}$ and gives a multilinear form in $(n-1)$ arguments. This form is denoted by $i_v\varphi$ and called an *inner product* of v and φ.

Exercise 4.6. Check that $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v, w_1, w_2, \dots, w_{n-1})$, i. e. the inner multiplication by v is just the fixation of v in the first argument.

4.7. Linear span of a tensor. Let $U, W \subset V$ be any two subspaces. Writing down the standard monomial bases, we see immediately that $(U \cap W)^{\otimes n} = U^{\otimes n} \cap W^{\otimes n}$ in $V^{\otimes n}$. So, for any $t \in V^{\otimes n}$ there is a *minimal* subspace $\mathrm{span}(t) \subset V$ whose n-th tensor power contains t. It is called *a linear span* of t and coincides with the intersection of all $W \subset V$ such that $t \in W^{\otimes n}$. To describe $\mathrm{span}(t)$ more constructively, for any injective (not necessary monotonous) map

$$J = (j_1, j_2, \dots, j_{n-1}) \, : \, \{1, 2, \dots, (n-1)\} \hookrightarrow \{1, 2, \dots, n\}$$

consider a linear map $V^{*\otimes(n-1)} \xrightarrow{c_t^J} V$ defined by complete contraction with t: it sends a decomposable tensor $\varphi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n-1}$ to a vector obtained by coupling ν-th factor ξ_ν, of φ, with j_ν-th factor of t for all $1 \leqslant \nu \leqslant (n-1)$, i.e.

$$c_t^J(\varphi) = c^{(1,2,\dots,(n-1))}_{(j_1, j_2, \dots, j_{n-1})}(\varphi \otimes t) \, .$$

4.7.1. CLAIM. *As a vector space,* $\mathrm{span}(t) \subset V$ *is linearly generated by the images* $c_t^J\left(V^{*\otimes(n-1)}\right)$ *taken for all possible* J.

Proof. Let $\mathrm{span}(t) = W \subset V$. Then $t \in W^{\otimes n}$ and $\mathrm{im}\left(c_t^J\right) \subset W \ \forall J$. It remains to prove that W is annihilated by any linear form $\xi \in V^*$ which annihilate all the subspaces $\mathrm{im}\left(c_t^J\right)$. Suppose the contrary: let $\xi \in V^*$ have non zero restriction on W but annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$. Then there exist a basis $\{w_1, w_2, \dots, w_k\}$ for W and a basis $\{\xi_1, \xi_2, \dots, \xi_d\}$ for V^* such that: $\xi_1 = \xi$, the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ onto W form the basis of W^* dual to $\{w_\nu\}$, and $\xi_{k+1}, \dots, \xi_d$ annihilate W. Now, for any J and $\xi_{i1}, \xi_{i2}, \dots, \xi_{in-1}$ we have

$$0 = \left\langle \xi \, , \, c_t^J\left(\xi_{i1} \otimes \xi_{i2} \otimes \cdots \otimes \xi_{in-1}\right) \right\rangle = \left\langle \xi_{i_{j_1^{-1}}} \otimes \cdots \otimes \xi_{i_{j_{s-1}^{-1}}} \otimes \xi \otimes \xi_{i_{j_{s+1}^{-1}}} \otimes \cdots \otimes \xi_{i_{j_n^{-1}}} \, , \, t \right\rangle \tag{4-4}$$

where $s = \{1, 2, \dots, n\} \setminus \mathrm{im}\,(J)$ and $J^{-1} = (j_1^{-1}, j_2^{-1}, \dots, j_n^{-1})$ is the inverse to J map from $\mathrm{im}\,(J) \subset \{1, 2, \dots, n\}$ to $\{1, 2, \dots, (n-1)\}$. Note that each basic monomial of $V^{*(n-1)}$ containing as a factor $\xi_1 = \xi$ can appear as the first operand in the right side of (4-4). But if we expand t trough the basic monomials $w_{i1} \otimes w_{i2} \otimes \cdots \otimes w_{in}$, then the coefficients of this expansion can be computed as full contractions of t with the corresponding elements $\xi_{i1} \otimes \xi_{i2} \otimes \cdots \otimes \xi_{in}$ from the dual basis for $W^{*\otimes n}$. By (4-4), such a contraction equals zero as soon one of ξ_{i_α} equals $\xi_1 = \xi$, which is dual to w_1. So, $\mathrm{span}(t)$ is contained in the linear span of $w_2, \dots, w_k$ but this contradicts our assumption. □

4.8. Symmetry properties. A multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ is called *symmetric* if it doesn't change its value under any permutations of the arguments. If the value of φ is stable under the even permutations and changes the sign under the odd ones, then φ is called *skew symmetric*. Since the composition operator (4-1) preserves the symmetry properties, for the (skew)symmetric φ the composition operator (4-1) turns into

$$\left(\begin{array}{c} \text{the space } \mathrm{Hom}(U, W) \text{ of all} \\ \text{linear operators } U \xrightarrow{F} W \end{array} \right) \xrightarrow{F \mapsto F \circ \varphi} \left(\begin{array}{c} \text{the space of all (skew)symmetric} \\ \text{multilinear maps } \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\psi} W \end{array} \right) \tag{4-5}$$

A (skew)symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} U$ is called *universal* if (4-5) is an isomorphism for any W. In the symmetric case the universal target space is denoted by S^nV and called n-th *symmetric power* of V. In the skew symmetric case it is called n-th *exterior power* of V and denoted by $\Lambda^n V$.

Exercise 4.7. Show that, if exist, $S^n V$ and $\Lambda^n V$ are unique up to unique isomorphism commuting with the universal maps.

4.9. Symmetric algebra $S^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a commutation relations $vw = wv$. More precisely, denote by $\mathscr{I}_{\text{sym}} \subset \mathsf{T}^\bullet V$ a linear span of all tensors

$$\cdots \otimes v \otimes w \otimes \cdots \quad - \quad \cdots \otimes w \otimes v \otimes \cdots ,$$

where the both terms are decomposable, have the same degree, and differ only in order of v, w. Clearly, $\mathscr{I}_{\text{sym}}$ is a double-sided ideal in $\mathsf{T}^\bullet V$ generated by a linear span of all the differences $v \otimes w - w \otimes v \in V \otimes V$. The factor algebra $S^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V / \mathscr{I}_{\text{sym}}$ is called *a symmetric algebra* of the vector space V. By the construction, it is commutative[1]. Since $\mathscr{I}_{\text{sym}} = \bigoplus_{n\geqslant 0} (\mathscr{I}_{\text{sym}} \cap V^{\otimes n})$ is the direct sum of its homogeneous components, the symmetric algebra is graded: $S^\bullet V = \bigoplus_{n\geqslant 0} S^n V$, where $S^n V \stackrel{\text{def}}{=} V^{\otimes n} / (\mathscr{I}_{\text{sym}} \cap V^{\otimes n})$.

4.9.1. CLAIM. *The tensor multiplication followed by the factorization map:*

$$\underbrace{V \times V \times \cdots \times V}_{n} \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} S^n(V) \tag{4-6}$$

gives the universal symmetric multilinear map.

Proof. *Any* multilinear map $V \times V \times \cdots \times V \xrightarrow{\varphi} W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $V^{\otimes n} \xrightarrow{F} W$ is linear. F is factored through π iff $F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots)$, i. e. iff $\varphi(\ldots, v, w, \ldots) = \varphi(\ldots, w, v, \ldots)$ □

The graded components $S^n V$ are called *symmetric powers* of V and the map (4-6) is called *a symmetric multiplication*. If a basis $\{e_1, e_2, \ldots, e_d\} \subset V$ is fixed, then $S^n V$ is naturally identified with the space of all homogeneous polynomials of degree n in e_i. Namely, consider the polynomial ring $k[e_1, e_2, \ldots, e_d]$ (whose «variables» are the basic vectors e_i) and identify V with the space of all linear homogeneous polynomials in e_i.

Exercise 4.8. Check that the multiplication map

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{(\ell_1, \ell_2, \ldots, \ell_n) \longmapsto \prod_{\nu=1}^{n} \ell_\nu} \begin{pmatrix} \text{the homogeneous polynomials} \\ \text{of degree } n \text{ in } e_i \end{pmatrix}$$

is universal and show that $\dim S^n V = \binom{d+n-1}{n}$.

4.10. Exterior algebra $\Lambda^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a skew commutation relations $vw = -wv$. More precisely, consider a double-sided ideal $\mathscr{I}_{\text{skew}} \subset \mathsf{T}^\bullet V$ generated by all sums $v \otimes w + w \otimes v \in V \otimes V$ and put $\Lambda^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V / \mathscr{I}_{\text{skew}}$. Exactly as in the symmetric case, the ideal $\mathscr{I}_{\text{skew}}$ is *homogeneous*: $\mathscr{I}_{\text{skew}} = \bigoplus_{n\geqslant 0} (\mathscr{I}_{\text{skew}} \cap V^{\otimes n})$, where $(\mathscr{I}_{\text{skew}} \cap V^{\otimes n})$ is the linear span of all sums

$$\cdots \otimes v \otimes w \otimes \cdots \quad + \quad \cdots \otimes w \otimes v \otimes \cdots$$

(the both items have degree n and differ only in the order of v, w). So, the factor algebra $\Lambda^\bullet V$ is graded by the subspaces $\Lambda^n V \stackrel{\text{def}}{=} V^{\otimes n} / (\mathscr{I}_{\text{skew}} \cap V^{\otimes n})$.

Exercise 4.9. Prove that the tensor multiplication followed by the factorization

$$\underbrace{V \times V \times \cdots \times V}_{n} \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} \Lambda^n(V) \tag{4-7}$$

[1] Again, if you like it, prove that for any commutative $\Bbbk$-algebra A and a vector space map $V \xrightarrow{f} A$ there exists a unique homomorphism of commutative algebras $S^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

gives the universal skew symmetric multilinear map.

The map (4-7) is called *an exterior* or *skew* multiplication. The skew product of vectors $(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. By the construction, it changes the sign under the transposition of any two consequent terms. So, under any permutation of terms the skew product is multiplied by the sign of the permutation.

Exercise 4.10. For any $U, W \subset V$ check that $S^n U \cap S^n W = S^n(U \cap U)$ in $S^n V$ and $\Lambda^n U \cap \Lambda^n W = \Lambda^n(U \cap U)$ in $\Lambda^n V$.

4.11. Grassmannian polynomials. Let $\{e_1, e_2, \dots, e_d\} \subset V$ be a basis . Then the exterior algebra $\Lambda^\bullet V$ is identified with *a grassmannian polynomial ring* $k\langle e_1, e_2, \dots, e_d\rangle$ whose «variables» are the basic vectors e_i which *skew commute*, that is, $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j. More precisely, it is linearly spanned by the grassmannian monomials $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. It follows from skew commutativity that $e_i \wedge e_i = 0$ for all i, that is, a grassmannian monomial vanishes as soon as it becomes of degree more then 1 in some e_i. So, any grassmannian monomial has a unique representation $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

4.11.1. CLAIM. *The monomials $e_I \overset{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, where $I = (i_1, i_2, \dots, i_n)$ runs through the increasing n-element subsets in $\{1, 2, \dots, d\}$, form a basis for $\Lambda^n V$. In particular, $\Lambda^n V = 0$ for $n > \dim V$, $\dim \Lambda^n V = \binom{d}{n}$, and $\dim k\langle e_1, e_2, \dots, e_d\rangle = 2^d$.*

Proof. Consider $\binom{d}{n}$-dimensional vector space U whose basis consists of the symbols ξ_I, where $I = (i_1, i_2, \dots, i_n)$ runs through the increasing n-element subsets in $\{1, 2, \dots, d\}$. Define a skew symmetric multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\alpha} U \ : \ (e_{j_1}, e_{j_2}, \dots, e_{j_n}) \longmapsto \operatorname{sgn}(\sigma) \cdot \xi_I \ ,$$

where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)},)$ is an *increasing* collection obtained from $(j_1, j_2, \dots, j_n)$ by a (unique) permutation σ. This map is universal. Indeed, for any skew symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ there exists at most one linear operator $U \xrightarrow{F} W$ such that $\varphi = F \circ \alpha$, because it has to act on the basis as $F(\xi_I) = \varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n}))$ for all increasing $I = (i_1, i_2, \dots, i_n)$. On the other side, such F really decomposes φ, because $F(\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n})) = \varphi(e_{j_1}, e_{j_2}, \dots, e_{j_n}))$ for all *not increasing* basis collections $(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \subset \underbrace{V \times V \times \cdots \times V}_{n}$ as well. By the universality, there exists a canonical isomorphism between U and $\Lambda^n V$ which sends ξ_I to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. □

Exercise 4.11. Check that

$$f(e) \wedge g(e) = (-1)^{\deg(f) \cdot \deg(g)} g(e) \wedge f(e)$$

for all homogeneous $f(e)$, $g(e) \in k\langle e_1, e_2, \dots, e_d\rangle$. In particular, each *even degree* homogeneous polynomial commutes with any grassmannian polynomial.

Exercise 4.12. Describe the center of $k\langle e_1, e_2, \dots, e_d\rangle$, i. e. all grassmannian polynomials which commute with everything.

4.11.2. Example: linear basis change in grassmannian polynomial. Under the linear substitution $e_i = \sum\limits_{j=1}^{d} a_{ij}\, \xi_j$ the basis monomials e_I are changed by the new basis monomials ξ_I as follows:

$$\begin{aligned} e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &= \Big(\sum_{j_1} a_{i_1 j_1}\, \xi_j\Big) \wedge \Big(\sum_{j_2} a_{i_2 j_2}\, \xi_j\Big) \wedge \cdots \wedge \Big(\sum_{j_n} a_{i_n j_n}\, \xi_j\Big) = \\ &= \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant n} \ \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \cdots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} = \sum_J a_{IJ}\, \xi_J \ , \end{aligned}$$

where a_{IJ} is $(n \times n)$-minor of (a_{ij}) placed at $(i_1, i_2, \dots, i_n)$ rows and $(j_1, j_2, \dots, j_n)$ columns, and J runs through all increasing index collections of *the length* $\#J = n$.

Exercise 4.13. Let $|I| = \sum\limits_{\nu} i_\nu$ denote *a weight* of the increasing index collection $I = (i_1, i_2, \dots, i_n)$ of length $\#I = n$. Check that

$$e_I \wedge e_{\widehat{I}} = (-1)^{|I| + \frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_d \tag{4-8}$$

for any two *complementary* index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$.

4.11.3. Example: the Sylvester relations via grassmannian polynomials. Let us take two complementary index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$ and do a basis change $e_i = \sum_{j=1}^{d} a_{ij}\, \xi_j$ in the identity (4-8). Its left side $e_I \wedge e_{\widehat{I}}$ turns to

$$\Bigg(\sum_{\substack{K: \\ \#K=\#I}} a_{IK}\, \xi_K \Bigg) \wedge \Bigg(\sum_{\substack{L: \\ \#L=(d-\#I)}} a_{L\widehat{I}}\, \xi_L \Bigg) = (-1)^{\frac{1}{2}\#I(1+\#I)} \sum_{\substack{K: \\ \#K=\#I}} (-1)^{|K|} a_{IK} a_{\widehat{I}\widehat{K}}\, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d \,,$$

where $\widehat{K} = \{1, 2, \dots, d\} \setminus K$. The right side of (4-8) gives $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|} \det(a_{ij}) \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d$. So, for any any collection I of rows in any square matrix (a_{ij}) the following relation holds:

$$\sum_{\substack{K: \\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} \widehat{a}_{IK} = \det(a_{ij}) \,, \tag{4-9}$$

where $\widehat{a}_{IK} \stackrel{\text{def}}{=} a_{\widehat{I}\widehat{K}}$ denotes the $(d-n) \times (d-n)$ - minor which is complementary[1] to a_{IK} and the summation runs over all $(n \times n)$ - minors a_{IK} contained in the rows $(i_1, i_2, \dots, i_n)$.

If we take $I \cap J \neq \varnothing$, then starting from $e_I \wedge e_J = 0$ instead of (4-8) we get by the same calculation the relation

$$\sum_{\substack{K: \\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} \widehat{a}_{JK} = 0 \,, \tag{4-10}$$

The identities (4-9) and (4-10) are known as *Sylvester relations*. Let us fix, say lexicographical, order on the set of indices I and arrange all $(n \times n)$-minors a_{IJ} as $\binom{d}{n} \times \binom{d}{n}$ - matrix $A^{(n)} \stackrel{\text{def}}{=} (a_{IJ})$. If we denote by $\widehat{A}^{(n)}$ a matrix whose (IJ)-entry equals $\big((-1)^{|I|+|J|}\widehat{a}_{JI}\big)$, then all the Sylvester relations are expressed by the single matrix equality $A^{(n)} \cdot \widehat{A}^{(n)} = \det(a_{ij}) \cdot E$.

4.11.4. Example: reduction of grassmannian quadratic forms. Any homogeneous grassmannian polynomial of degree 2 can be written as

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots + \xi_{r-1} \wedge \xi_r \tag{4-11}$$

in some basis (over any field $\Bbbk$). Namely, we can suppose[2] that our grassmannian quadratic form is

$$q(e) = e_1 \wedge (\alpha_2 e_2 + \cdots + \alpha_n e_n) + (\text{terms without } e_1)$$

where $\alpha_2 \neq 0$ and $\xi_2 \stackrel{\text{def}}{=} \alpha_2 e_2 + \cdots + \alpha_n e_n$ does not contain e_1, that is it can be included in the new basis $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ with $\zeta_i = e_i$ for $i \neq 2$. After the substitution $e_2 = \alpha_2^{-1}(\xi_2 - \alpha_3\zeta_3 - \cdots - \alpha_n\zeta_n)$, $e_i = \zeta_i$ for $i \neq 2$, we can write q as $q(\zeta) = \zeta_1 \wedge \zeta_2 + \zeta_2 \wedge (\beta_3\zeta_3 + \cdots + \beta_n\zeta_n) + (\text{terms without } \zeta_1 \text{ and } \zeta_2)$. So, in the next new base: $\{\xi_1, \xi_2, \dots, \xi_n\}$ with $\xi_1 = \zeta_1 - \beta_3\zeta_3 - \cdots - \beta_n\zeta_n$, $\xi_i = \zeta_i$ for $i \neq 1$ our q turns to

$$q(\zeta) = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1 \text{ and } \xi_2)$$

and this procedure can be repeated inductively for the remaining terms.

Exercise 4.14. Let $A = (a_{ij})$ be a skew symmetric matrix (i. e. $a_{ij} = -a_{ji}$) and $q(e) = \sum_{ij} a_{ij}\, e_i \wedge e_j$ be a grassmannian quadratic form. Show that in the representation (4-11) the number r doesn't depend on the basis choice and equals $\operatorname{rk} A$. (In particular, $\operatorname{rk} A$ is always even.)

[1] i. e. sitting in the complementary rows and columns
[2] may be after appropriate renumbering of the basic vectors

§5. Polarizations and contractions.

In this section we always assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk \neq 2$.

5.1. (Skew) symmetric tensors. A symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ permuting factors in the decomposable tensors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \overset{\text{def}}{=} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in \mathfrak{S}_n$. Subspaces

$$\begin{gathered}\mathsf{AT}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = \operatorname{sgn}(\sigma) \cdot t \quad \forall\, \sigma \in \mathfrak{S}_n \,\} \\ \mathsf{ST}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}\end{gathered}$$

are called the spaces of *skew symmetric* and *symmetric* tensors.

5.1.1. CLAIM. *Let* $\operatorname{char}(k) = 0$*. Restricting the canonical factorization maps*

$$V^{\otimes n} \xrightarrow{\pi_{skew}} \Lambda^n V\,, \quad V^{\otimes n} \xrightarrow{\pi_{sym}} S^n V\,,$$

onto the spaces of (skew) symmetric tensors, we get the isomorphisms

$$\mathsf{AT}^n V \xrightarrow{\pi_{skew}} \Lambda^n V \qquad \textit{and} \qquad \mathsf{ST}^n V \xrightarrow{\pi_{sym}} S^n V\,. \tag{5-1}$$

Proof. In the skew symmetric case, a basis of $\mathsf{AT}^n V$ is formed by the tensors

$$e_{\langle i_1, i_2, \dots, i_n \rangle} \overset{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}$$

(sum of all the tensor monomials sent to the basic Grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ by π_{skew}). So, $\pi_{\text{skew}}\left(e_{\langle i_1, i_2, \dots, i_n \rangle}\right) = n!\, e_I$. In the symmetric case, let us write $e_{[m_1, m_2, \dots, m_d]}$ for the sum of all tensor monomials containing m_1 factors e_1, m_2 factors e_2, … , m_d factors e_d, where $\sum\limits_{\nu} m_\nu = n$. These monomials form one $\mathfrak{S}_n$-orbit, which consists of $\frac{n!}{m_1!\, m_2! \cdots m_d!}$ elements and collects all the decomposable tensors sent to $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ by π_{sym}. As above, the tensors $e_{[m_1, m_2, \dots, m_d]}$ form a basis for $\mathsf{ST}^n V$ and $\pi_{\text{sym}}\left(e_{[i_1, i_2, \dots, i_n]}\right) = \frac{n!}{m_1!\, m_2! \cdots m_d!} e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. □

Exercise 5.1. Verify that the above sums $e_{\langle i_1, i_2, \dots, i_n \rangle}$ and $e_{[m_1, m_2, \dots, m_d]}$ really give the bases for $\mathsf{ST}^n V$ (over any field of an arbitrary characteristic). Also note that if $\operatorname{char}(k) > 0$ divides n, then all these basic (skew) symmetric tensors are annihilated by factorization through (skew) symmetric relations.

Exercise 5.2. Verify that if $\operatorname{char}(k) = 0$, then $V^{\otimes n} = \mathscr{I}^{(n)}_{\text{skew}} \oplus \mathsf{AT}^n V = \mathsf{ST}^n V \oplus \mathscr{I}^{(n)}_{\text{sym}}$, where the projection $V^{\otimes n} \twoheadrightarrow \mathsf{ST}^n V$ along $\mathscr{I}^{(n)}_{\text{sym}}$ is given by *the symmetrization map*

$$\operatorname{sym}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(\tau)$$

and the projection $V^{\otimes n} \twoheadrightarrow \mathsf{AT}^n V$ along $\mathscr{I}^{(n)}_{\text{skew}}$ given by *the alternation map*

$$\operatorname{alt}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \sigma(\tau)\,.$$

5.2. Polarization of (skew) polynomials. The inverse maps to the isomorphisms (5-1) take

$$\begin{gathered} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \longmapsto \frac{1}{n!} \cdot e_{\langle i_1, i_2, \dots, i_n \rangle} \\ e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \longmapsto \frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot e_{[m_1, m_2, \dots, m_d]}\,. \end{gathered} \tag{5-2}$$

The both maps are called *complete polarizations* of (skew) polynomials and are denoted by $f \longmapsto \operatorname{pl}(f)$.

5.2.1. Example: (skew) polynomials and (skew) symmetric multilinear forms. Full polarization $\operatorname{pl}(f)$ of a (skew) homogeneous degree n polynomial f of one argument on V can be considered as a multilinear form of n arguments on V. It sends $(v_1, v_2, \dots, v_n)$ to the full contraction $\widetilde{f}(v_1, v_2, \dots, v_n) \stackrel{\text{def}}{=} \langle\, v_1 \otimes v_2 \otimes \, \cdots \, \otimes v_n \,,\, \mathrm{pl}(f) \,\rangle$ and has the same symmetry properties as f, because for all $\sigma \in \mathfrak{S}_n$, $t \in V^{\otimes n}$, $\xi \in V^{*\otimes n}$ we have $\langle\, \sigma(t) \,,\, \sigma(\xi) \,\rangle = \langle\, t \,,\, \xi \,\rangle$, which implies $\langle\, \sigma(t) \,,\, \xi \,\rangle = \langle\, t \,,\, \sigma^{-1}(\xi) \,\rangle$.

Exercise 5.3. Check that for a symmetric quadratic form $q(x) \in S^2V^*$ we have

$$\widetilde{q}(x,y) = \frac{q(x+y) - q(x-y)}{4} = \frac{q(x+y) - q(x) - q(y)}{2} = \frac{1}{2} \sum_{\nu=1}^{\dim V} y_\nu \frac{\partial q}{\partial x_\nu} \,.$$

Since any multilinear form φ may be presented via full contraction

$$\varphi(v_1, v_2, \dots, v_n) = \langle\, v_1 \otimes v_2 \otimes \, \cdots \, \otimes v_n \,,\, \xi \,\rangle$$

with some $\xi \in V^{*\otimes n}$, the complete polarization identifies S^nV^* and Λ^nV^* with the spaces of all symmetric and skew symmetric multilinear forms $\underbrace{V \times V \times \, \cdots \, \times V}_{n} \longrightarrow \Bbbk$, in n arguments on V.

Exercise 5.4. In symmetric case, show that a homogeneous degree n polynomial $f(x)$ (in one argument $x \in V$) coincides with the restriction of the corresponding symmetric multilinear form $\widetilde{f}(x_1, x_2, \dots, x_n)$ (in n arguments $x_\nu \in V$) onto the diagonal: $f(x) = \widetilde{f}(x, x, \, \dots \, , x)$.

5.2.2. Example: duality on polynomials. Using the complete polarization and the full contraction between $V^{\otimes n}$ and $V^{*\otimes n}$ we obtain (over a field of zero characteristic) a natural non degenerate pairing between $\Lambda^n(V)$ and $\Lambda^n\,(V^*)$ as well as between $S^n(V)$ and $S^n\,(V^*)$. Namely, for two (skew) polynomials f, in $e_i \in V$, and ξ, in $x_i \in V^*$, we put $\langle\, f \,,\, \xi \,\rangle \stackrel{\text{def}}{=} \langle\, \mathrm{pl}(f) \,,\, \mathrm{pl}(\xi) \,\rangle$.

Exercise 5.5. Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{x_1, x_2, \dots, x_n\} \subset V^*$ be dual bases. Check that:

$$\langle\, e_I \,,\, x_J \,\rangle = \begin{cases} 1/n! & \text{, for } I = J \\ 0 & \text{, for } I \neq J \end{cases} \tag{5-3}$$

$$\langle e_1^{m_1} e_2^{m_2} \, \cdots \, e_d^{m_d} \,,\, x_1^{\ell_1} x_2^{\ell_2} \, \cdots \, x_d^{\ell_d} \rangle = \begin{cases} \frac{m_1!\, m_2! \cdots m_d!}{n!} & \text{, if } m_\nu = \ell_\nu \ \forall \nu \\ 0 & \text{, otherwise} \end{cases} \tag{5-4}$$

5.3. Partial derivatives (symmetric case). For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ does not depend on the choice of contracted index i in $\mathrm{pl}(f)$, because $\mathrm{pl}(f) \in \mathsf{ST}^nV^*$ is symmetric. Its projection to $S^{n-1}V^*$ is called *a polar* of v w.r.t. f and is denoted by $\mathrm{pl}_v f$.

Exercise 5.6. Show that $\deg(f) \cdot \mathrm{pl}_v f = \partial_v f$, where ∂_v is the derivative in v-direction, which takes f to $\sum\limits_{i=1}^{\dim V} v_i \, \frac{\partial f}{\partial x_i}$ (here $v = \sum v_i e_i$, $\{e_i\}$ is a basis for V and $\{x_i\}$ is a dual basis for V^*).

Exercise 5.7. Show that $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \partial_{v_1} \partial_{v_2} \, \cdots \, \partial_{v_n} \, f$ for any $f \in S^nV^*$, $v_1, v_2, \dots, v_n \in V$.

It follows that $\partial_v f$ does not depend on a choice of basis, is bilinear in v, f, and satisfies

$$m_2!\, \partial_{v_1}^{m_1} f(v_2) = (m_1 + m_2)!\, \widetilde{f}(\underbrace{v_1, v_1, \, \dots \, , v_1}_{m_1}, \underbrace{v_2, v_2, \, \dots \, , v_2}_{m_2}) = m_1!\, \partial_{v_2}^{m_2} f(v_1) \,, \tag{5-5}$$

where $(m_1 + m_2) = n = \deg f$. In particular, the left and right sides are bihomogeneous of bidegree (m_1, m_2) in (v_1, v_2).

Exercise 5.8. Show that *multiple polars*: $\mathrm{pl}_{v_1, v_2, \dots, v_m} f \stackrel{\text{def}}{=} \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \, \cdots \, \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f, that is come from a linear map $S^mV \otimes S^nV^* \longrightarrow S^{n-m}V^*$, which sends $v_1 v_2 \, \cdots \, v_m \otimes f$ to $\mathrm{pl}_{v_1, v_2, \dots, v_m} f(w) = \widetilde{f}(v_1, v_2, \dots, v_m, w, w, \, \dots \, , w)$.

5.3.1. Example: the Taylor formula. For $f \in S^nV^*$ the value $f(v+w)$ can be computed as the full contraction of f with $(v+w)^m = \sum\limits_m \binom{n}{m} v^m w^{n-m} \in S^nV$. This can be arranged as the *Taylor formula*:

$$f(v+w) = \sum_{\nu=0}^{n} \frac{1}{\nu!} \, \partial_w^\nu f(v) \,. \tag{5-6}$$

5.3.2. Example: $\mathrm{span}(f)$ for $f \in S^nV^*$, that is a minimal subspace $W \subset V^*$ such that $f \in S^nW$, coincides with $\mathrm{span}(\mathrm{pl}(f))$ described in n° 4.7.1 as an image of the contraction map $S^{n-1}V \otimes S^nV^* \longrightarrow V^*$. So, in terms of partial derivatives, $\mathrm{span}(f)$ is generated by the linear forms

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{5-7}$$

obtained from f by all possible $(n-1)$-fold differentiations.

5.4. Veronese variety in $\mathbb{P}_N = \mathbb{P}(S^nV^*)$, where $\dim V = d+1$, $N = \binom{n}{d} - 1$, consists of pure n-th powers of linear forms $\xi \in \mathbb{P}(V^*)$. It has a rational parameterization given by the Veronese map

$$\mathbb{P}(V^*) \xhookrightarrow{\xi \longmapsto \xi^n} \mathbb{P}(S^nV^*) \,.$$

The result of n° 5.3.2 allows to present the Veronese variety by a system of quadratic equations. Namely, $f \in S^nV^*$ equals $\xi^n(x)$ for some $\xi \in V^*$ iff $\mathrm{span}(\widetilde{f}) \subset V^*$ coincides with the 1-dimensional subspace generated by ξ. So, f is pure n-th power iff all the linear forms (5-7) are proportional to each other. If we arrange the coefficients of these forms in the rows of some $2\times(d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

5.4.1. Example: Veronese's curve. Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^nU^*)$, $\{t_0, t_1\}$ be a basis of U^*, and $\binom{n}{i} t_0^i t_1^{n-i}$ for $0 \leqslant i \leqslant n$ be the corresponding basis of S^nV^*. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

Its image is a rational curve $C_n \subset \mathbb{P}_n$ called *Veronese curve* or *rational normal curve* of degree n. If we use the coefficients $(a_0 : a_1 : \ \dots \ : a_n)$ and $(\alpha_0 : \alpha_1)$ as homogeneous coordinates for a polynomial $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^nU^*$ and for a linear form $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$, then C_n will be presented parametrically as $a_i = \alpha_0^i \alpha_1^{n-i}$.

On the other hand, for $f(t) = \sum\limits_i a_i \cdot \dbinom{n}{i} t_0^i t_1^{n-i}$ the linear forms (5-7) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \,, \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \,.$$

So, $f \in C_n \iff \mathrm{rk} \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1 \iff a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$.

In particular, for $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercise 5.9. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0:0:0:1)$ and at $(1:0:0:0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!

5.5. Partial derivatives (skew-symmetric case). The contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ between a vector $v \in V$ and a skew polynomial $f \in \Lambda^nV^*$ slightly depends on a choice of the contracted index i: it changes the sign when i is incremented or decremented by one, because $\mathrm{pl}(f) \in \mathsf{AT}^nV^*$ is skew symmetric. Let us choose $i = 1$ and write $\mathrm{pl}_v f$ for the projection of $c_1^1(\mathrm{pl}(f) \otimes v)$ into $\Lambda^{n-1}V^*$.

Since the consecutive polarizations defined by this rule do *anticommute*: $\mathrm{pl}_v \mathrm{pl}_w f = -\mathrm{pl}_w \mathrm{pl}_v f$, the multilinear map

$$\underbrace{V \times V \times \cdots \times V}_{m} \times \Lambda^nV^* \xrightarrow{(v_1, v_2, \dots, v_m, f) \mapsto \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_m} f} \Lambda^{n-m}V^*$$

comes from the linear map $\Lambda^mV \otimes \Lambda^nV^* \longrightarrow \Lambda^{n-m}V^*$.

Exercise 5.10. Show that $\deg f \cdot \mathrm{pl}_v f = \partial_v f$, where $\partial_v = \sum v_i \frac{\partial}{\partial x_i}$, as in ex. 5.6, and verify the following properties of skew partial derivatives: a) $\partial_v \partial_w = -\partial_w \partial_v$ (in particular, $\partial_v^2 = 0 \quad \forall v \in V$);
b) $\widetilde{f}(v_1, v_2, \dots, v_n) = \frac{1}{n!}\, \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in \Lambda^n V^*$, $v_1, v_2, \dots, v_n \in V$.
c) $\partial_v(f_1 \wedge f_2) = (\partial_v f_1) \wedge f_2 - (-1)^{\deg f_1} f_1 \wedge (\partial_v f_2)$, in particular,

$$\partial_{e_{i_\nu}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = (-1)^{\nu-1} x_{i_1} \wedge \cdots \wedge x_{i_{\nu-1}} \wedge x_{i_{\nu+1}} \wedge \cdots \wedge x_{i_n} \; .$$

5.5.1. Example: $\mathrm{span}(f)$ for grassmannian polynomial $f \in \Lambda^n V$ is linearly generated by partial derivatives

$$\partial_J f = \partial_{x_{j_{n-1}}} \partial_{x_{j_{n-2}}} \dots \partial_{x_{j_1}} f = n! \left\langle x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_{n-1}} \, , \, \mathrm{pl}(f) \right\rangle$$

where $J = (j_1, j_2, \dots, j_{n-1}) \subset \{1, 2, \dots, d\}$ runs through all $\binom{d}{n-1}$ increasing ordered collections of $(n-1)$ indexes and $\{x_i\} \in V^*$ form a dual basis to some basis $\{e_i\} \in V$. If $f = \sum\limits_I \alpha_I \, e_I$ w.r.t. the last base, then

$$\partial_J f = n! \cdot \sum_{i \notin J} (-1)^{n-p(i,J)} \, \alpha_{J \sqcup \{i\}} \, e_i \tag{5-8}$$

where $p(i, J)$ is the number of place where i stays in the increasing permutation I of $J \sqcup \{i\}$ (because x_I is the only monomial in $\Lambda^n V^*$ whose complete contraction with e_I is non zero, see (5-3)).

5.5.2. Example: Plücker relations. A skew polynomial $f \in \Lambda^n V$ is called *completely decomposable*, if $f = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is a product of n linear factors, or equivalently, if $\dim \mathrm{span}(f) = n$ is minimal possible.

Exercise 5.11. Show that f is completely decomposable iff $f \wedge v = 0 \; \forall v \in \mathrm{span}(f)$.

So, like in n° 5.3.2 — n° 5.4.1, the set of completely decomposable polynomials $f \in \Lambda^n V$ is described by a system of quadratic equations $(\partial_J f) \wedge f = 0$. By (5-8), a basic monomial $e_K \in \Lambda^{n+1} V$ appears in

$$(\partial_J f) \wedge f = n! \cdot \left(\sum_{i \notin J} (-1)^{p(i,J)} \, \alpha_{J \sqcup \{i\}} \, e_i \right) \wedge \left(\sum_I a_I e_I \right)$$

as $n! \cdot \sum\limits_{\alpha=1}^{n+1} (-1)^{n-p(k_\alpha, J)} \cdot \alpha_{J \sqcup \{k_\alpha\}} \alpha_{K \setminus \{k_\alpha\}} \cdot e_{k_\alpha} \wedge e_{K \setminus \{k_\alpha\}}$, i.e. with the coefficient, which up to a constant factor equals

$$P_{JK}(f) \stackrel{\text{def}}{=} \sum_{i \in K \setminus (K \cap J)} (-1)^{p(i,J,K)} \, \alpha_{J \sqcup i} \, \alpha_{K \setminus i} \; , \tag{5-9}$$

where $p(i, J, K)$ is the sum of the place numbers where i stays in the increasing version of $J \sqcup \{i\}$ and in K.

Exercise 5.12. Show that this coefficient vanishes, if $J \subset K$

System of quadratic equations $P_{JK}(f) = 0$, which defines the variety of completely decomposable Grassmannian polynomials, is known as the *Plücker relations*. Note that these quadratic equations are not «independent», even not pairwise different (see n° 6.6.2 below).

§6. Working example: grassmannians.

6.1. Plücker quadric in $\mathbb{P}_5$. Let V be a 4-dimensional vector space and $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Then the *Plücker quadric*

$$Q_P \overset{\text{def}}{=} \{\, \omega \in \Lambda^2 V \mid \omega \wedge \omega = 0 \,\}$$

is a non singular quadric in $\mathbb{P}_5$. Fixing a base $\{e_0, e_1, e_2, e_3\}$ for V and induced base $e_{ij} = e_i \wedge e_j$ for $\Lambda^2 V$ and writing x_{ij} for the homogeneous coordinate along e_{ij}, we have $\Big(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) \wedge \Big(\sum\limits_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) = 2\,(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$, i.e. Q_P has the equation

$$x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}\ .$$

In coordinateless terms, Q_P is given by quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$ whose polarization $\widetilde{q}(\omega_1, \omega_2)$ is a bilinear form on $\Lambda^2 V$ defined up to a scalar factor by the prescription

$$\omega_1 \wedge \omega_2 = \widetilde{q}(\omega_1, \omega_2) \cdot \Omega\ ,$$

where $\Omega \in \Lambda^4 V \simeq \Bbbk$ is any fixed non-zero vector. This form is symmetric, because even degree Grassmannian polynomials commute: $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$.

6.2. Plücker embedding. By the definition, *a grassmannian* $\mathrm{Gr}(2,4)$ is a set of all lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$, or equivalently, a set of all 2-dimensional vector subspaces $U \subset V$. *A Plücker map*

$$\mathrm{Gr}(2,4) \overset{\mathfrak{u}}{\hookrightarrow} \mathbb{P}(\Lambda^2 V)$$

sends a 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$. If U is spaned by a pair of vectors u_1, u_2 (i.e. $\ell = \mathbb{P}(U)$ pass trough $u_1, u_2 \in \mathbb{P}(V)$), then $\mathfrak{u}(\ell) = \mathfrak{u}(U) = u_1 \wedge u_2$ up to proportionality.

6.2.1. LEMMA. *Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting iff* $\quad \widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = \mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = 0.$

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and there exist a base $\{e_i\} \subset V$ such that U_1 is spaned by e_0, e_1 and U_2 is spaned by e_2, e_3. So, $\mathfrak{u}(U_1) = e_0 \wedge e_1$, $\mathfrak{u}(U_2) = e_2 \wedge e_3$ and $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $U_1 \cap U_2 \neq 0$, then taking $u_0 \subset U_1 \cap U_2$ we can write $\mathfrak{u}(U_1) = u_0 \wedge u_1$, $\mathfrak{u}(U_2) = u_0 \wedge u_2$ for some u_1, u_2. So, $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = u_0 \wedge u_1 \wedge u_0 \wedge u_2 = 0$. □

6.2.2. LEMMA. *If $\dim V = 4$, then $\omega \in \Lambda^2 V$ is decomposable*[1] *iff* $\quad \omega \wedge \omega = 0.$

Proof. If ω is decomposable, say $\omega = u_1 \wedge u_2$, then $\omega \wedge \omega = u_1 \wedge u_2 \wedge u_1 \wedge u_2 = 0$, because of skew symmetry. To get the inverse, take a base $\{\xi_i\}$ such that ω turns into either $\omega = \xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3$ or $\omega = \xi_0 \wedge \xi_1$. In the first case $\omega \wedge \omega = 2\,\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \neq 0$, i.e. ω is indecomposable. □

6.2.3. COROLLARY. *The Plücker map is a bijection between the grassmannian $\mathrm{Gr}(2,4)$ and the Plücker quadric $Q_P \subset \mathbb{P}_5$.*

Proof. For any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersect ℓ_1 and doesn't intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$ imply $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$, i.e. $\mathfrak{u}$ is injective. Surjectivity follows from n° 6.2.2. □

6.2.4. COROLLARY. *For any point $p = \mathfrak{u}(\ell) \in Q_P$ the intersection $Q_P \cap T_p Q_P$ consists of all $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$.*

Proof. $T_p Q_P$ is a zero set of the linear form $\widetilde{q}(\mathfrak{u}(\ell), *)$. By n° 6.2.1, $\widetilde{q}(\mathfrak{u}(\ell), \mathfrak{u}(\ell')) = 0 \iff \ell \cap \ell' \neq \varnothing$. □

6.3. Line nets and line pencils in $\mathbb{P}_3$. A set of lines on $\mathbb{P}_3$ is called *a net* if it is represented by a plane $\pi \subset Q_P \subset \mathbb{P}_5$. If $\pi \subset Q_P$ is spaned by 3 non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, then

$$\pi = Q_P \cap T_{p_1} Q_P \cap T_{p_2} Q_P \cap T_{p_3} Q_P\ .$$

[1] recall that homogeneous polynomial is called decomposable if it is factorized into a product of linear forms

So, by n° 6.2.1 and n° 6.2.4 the corresponding line net consist of all lines which intersect 3 given pairwise intersecting lines. Hence, there are exactly two geometrically different line nets on $\mathbb{P}_3$:

α-net is a set of all lines passing through a given point $O \in \mathbb{P}_3$; the corresponding plain $\pi_\alpha(O) \subset Q_P$ is called *α-plane*. It is spanned by the Plücker images of any 3 non coplanar lines passing through O.

β-net is a set of all lines laying on a given plane $\Pi \in \mathbb{P}_3$; the corresponding plain $\pi_\beta(\Pi) \subset Q_P$ is called *β-plane*. It is spanned by the Plücker images of any 3 lines which lay on Π and don't have a common intersection.

Note that any two planes of the same type have exactly 1-point intersection, namely:

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\,\Pi_1 \cap \Pi_2\,)$$
$$\pi_\alpha(O_1) \cap \pi_\alpha(O_2) = \mathfrak{u}(\,(O_1 O_2)\,)$$

Two planes of different types $\pi_\beta(\Pi)$, $\pi_\alpha(O)$ do not intersect each other, if $O \notin \Pi$. If $O \in \Pi$, then $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ is a pencil of lines $\ell \subset \mathbb{P}_3$ such that $O \in \ell \subset \Pi$.

Exercise 6.1. Show that there are no other line pencils in $\mathbb{P}_3$, i. e. each line on $Q_P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ for some O and Π.

HINT. Consider the cone $C = Q_P \cap T_p Q_P$. It has a vertice at p and consists of all lines which pass through p and lay on Q_p. Fix a 3-dimensional hyperplane $H \subset T_p Q_P$ which doesn't contain p. Then $G = C \cap H$ is non singular quadric on H. So, any line passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and the planes π_α, π_β are spaned by p and two lines passing through p' on G (see fig 6◇1).

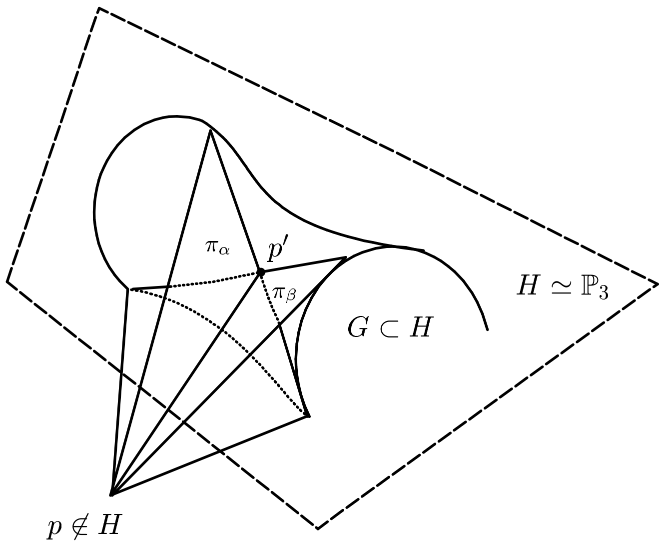


Fig 6◇1. The cone $C = Q_P \cap T_p Q_P$.

6.4. Affine cell decomposition of Gr(2, 4). Let $H \subset T_p Q_P$ be a 3-dimensional projective hyperplane such that $p \notin H$, as in above exercise, $C = Q_P \cap T_p Q_P$, and $G = H \cap Q_P$. Then C is the simple cone with vertex p over G (see fig 6◇1) and we have the following diagram of inclusions

$$\begin{tikzcd}
 & & \pi_\alpha \arrow[dr, hook] & & \\
p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & Q_P \\
 & & \pi_\beta \arrow[ur, hook] & &
\end{tikzcd}
\qquad \Rightarrow \qquad
\mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4$$

The right side decomposition is produced via replacing each stratum in the left side by the complement to all the smallest strata it contain and identifying the resulting disjoint cells with affine spaces as follows: $(\pi_\alpha \cap \pi_\beta) \backslash p \simeq \mathbb{A}^1$ (because this is a projective line without a point), $\pi_\alpha \backslash (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \backslash (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2$ (because the both are projective planes without a line), $C \backslash (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times \Big(G \setminus \big(G \cap T_{p'} G\big)\Big)$ (because C is a cone over G), and finally, $G \setminus \big(G \cap T_{p'} G\big) \simeq \mathbb{A}^2$ and $Q \setminus C \simeq \mathbb{A}^4$, because of the lemma below.

6.4.1. LEMMA. *Let $Q \subset \mathbb{P}_n$ be a quadric, $p \in Q$ be a non singular point, and $H \not\ni p$ be a codimension 1 hyperplane. Then the projection from p onto H induces a bijection between $Q \setminus (Q \cap T_p Q)$ and $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$.*

Proof. Any non tangent line passing through p have to intersect Q precisely ones more. All such lines are 1–1 parameterized by the points of $\mathbb{A}^{n-1} = H \setminus (H \cap T_p Q)$. ☐

Exercise 6.2*. If you have some experience in topology, show that over $\mathbb{C}$ all odd integer homologies of Gr(2,4) vanish and the even ones are $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$. Also, try to compute the homologies for the real grassmannian, where the boundary maps are non trivial.

6.5. General grassmannian $\mathbf{Gr}(m,d)$ is defined as the set of all m-dimensional vector subspaces in a given d-dimensional vector space V. If the nature of V is important, we write $\mathrm{Gr}(m,V)$ instead of $\mathrm{Gr}(m,d)$. In the projective language, $\mathrm{Gr}(m,d)$ is the set of all $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. If $m=1$, then $\mathrm{Gr}(m,d)=\mathbb{P}_{d-1}$. There is a canonical bijection $\mathrm{Gr}(m,V)\simeq\mathrm{Gr}(d-m,V^*)$ induced by duality. It sends $U\subset V$ to $\mathrm{Ann}\,U\subset V^*$ and wise versa.

Exercise 6.3. Let $\dim V=4$. Fix an isomorphism $V\xrightarrow{\widehat{q}}V^*$, say presented by a non singular quadric $Q\subset\mathbb{P}(V)$, and consider an automorphism of $\mathrm{Gr}(2,V)$ given by $U\longmapsto\mathrm{Ann}\,\widehat{q}(U)$. Show that it maps the α-planes on $\mathrm{Gr}(2,4)$ to the β-planes and wise versa.

6.6. Plücker embedding $\mathbf{Gr}(m,V)\overset{\mathfrak{u}}{\hookrightarrow}\mathbb{P}(\Lambda^m V)$ sends m-dimensional subspace $U\subset V$ to the 1-dimensional subspace $\Lambda^m U\subset\Lambda^m V$. If U is based by the vectors $\{u_1,u_2,\dots,u_m\}\subset U$, then $\mathfrak{u}(U)=u_1\wedge u_2\wedge\cdots\wedge u_m$ up to proportionality, because taking an other base, say $v_i=\sum a_{ij}u_j$, we get

$$\mathfrak{u}(U)=v_1\wedge v_2\wedge\cdots\wedge v_m=\det(a_{ij})\cdot u_1\wedge u_2\wedge\cdots\wedge u_m\,.$$

6.6.1. LEMMA. *The Plücker embedding is really injective.*

Proof. If $U_1\neq U_2$, then there exist a base in V such that some vectors $w_1,w_2,\dots,w_r$ of this base give a base for $U_1\cap U_2$, some other $u_1,u_2,\dots,u_{m-r}$ together with $\{w_\nu\}$ give a base for U_1, some other $v_1,v_2,\dots,v_{m-r}$ together with $\{w_\nu\}$ give a base for U_2, and the rest $e_1,e_2,\dots,e_{d+r-2m}$ are complementary to U_1+U_2. Let $\omega\in\Lambda^{d-m}V$ be the skew product of all v_ν and e_ν. The skew multiplication by ω $\Lambda^m V\xrightarrow{\xi\longmapsto\xi\wedge\omega}\Lambda^d V\simeq\Bbbk$ turns into a linear form on $\Lambda^m V$ as soon as a base vector for $\Lambda^d V$ is fixed. This linear form does vanish at $\mathfrak{u}(U_2)$ and doesn't at $\mathfrak{u}(U_1)$. $\square$

6.6.2. Example: 2×2-minors of 2×4-matrices. Let $\dim V=4$ and a base $\{e_1,e_2,e_3,e_4\}\subset V$ be fixed. Then a subspace $U\subset V$ based by u_1,u_2 can be presented as 2×4-matrix $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\end{pmatrix}$ whose rows are the coordinates of u_1,u_2. This matrix is defined by U up to the left multiplication $A\mapsto C\cdot A$ by any $C\in\mathrm{GL}_2(k)$ (this corresponds to a base change in U). The Plücker embedding sends A to

$$u_1\wedge u_2=\sum_{i<j}\det\begin{pmatrix}a_{1i}&a_{1j}\\a_{2i}&a_{2j}\end{pmatrix}e_i\wedge e_j\,.$$

So, the homogeneous coordinates of $\mathfrak{u}(U)\in\mathbb{P}_5$ in the base $\{e_{ij}=e_i\wedge e_j\}$ are six 2×2-minors of A. In particular, the left multiplications by $C\in\mathrm{GL}_2$ doesn't effect on the ratios between 2×2-minors of A. An other claim: six numbers $x_1,x_2,\dots,x_6$ give a collection of 2×2-minors for some 2×4 matrix iff they satisfy (maybe, after some renumbering) the Plücker equation $x_1x_2=x_3x_4+x_5x_6$.

Exercise 6.4. Is there 2×4 - matrix with minors a) $\{2,3,4,5,6,7\}$ b) $\{3,4,5,6,7,8\}$?

6.7. Matrix notations and Plücker coordinates on $\mathbf{Gr}(m,d)$. A point $U\in\mathrm{Gr}(m,d)$ can be presented by $(m\times d)$-matrix A_U whose rows are the coordinates of some base vectors $\{u_1,u_2,\dots,u_m\}\subset U$ with respect to a fixed base $\{e_1,e_2,\dots,e_d\}\subset V$. Such a matrix is not unique and is defined by U only up to the left multiplication by any $C\in\mathrm{GL}_m$ (this corresponds to a base change in U). So, the grassmannian $\mathrm{Gr}(m,d)$ can be considered as a factor space of $\mathrm{Mat}_{m\times d}(k)$ by the left action of $\mathrm{GL}_m(k)$. Under the Plücker embedding , the homogeneous coordinates of $\mathfrak{u}(U)\in\Lambda^m V$ in the standard base $e_I=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}$ are equal to the maximal minors of A. They are stable under the left GL_m-action and are called *Plücker coordinates* of U.

6.8. Affine covering and affine coordinates on $\mathbf{Gr}(m,d)$. Consider the standard affine card $\mathscr{U}_I\subset\mathbb{P}(\Lambda^m V)$ given by $x_I=1$, where x_I is the coordinate along $e_I=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}$. The inverse image $\mathfrak{U}_I\overset{\text{def}}{=}\mathfrak{u}^{-1}(\mathscr{U}_I)\subset\mathrm{Gr}(m,d)$ consists of all U such that A_U has non zero maximal minor in the columns $(i_1,i_2,\dots,i_m)$. Any such U has *a unique* matrix representation $A_U^{(I)}=\left(a_{ij}^{(I)}\right)$ with *the identity $m\times m$-submatrix* staying in these columns. It is given by $A_U^{(I)}=A_{U,I}^{-1}\cdot A_U$, where A_U is an arbitrary matrix representation for U and $A_{U,I}\subset A_U$ is $m\times m$-submatrix formed by the columns $(i_1,i_2,\dots,i_m)$. So, the points of $\mathfrak{U}_I\subset\mathrm{Gr}(m,d)$ are 1–1 parametrized by $m\,(d-m)$ matrix elements $\left(a_{\mu\nu}^{(I)}\right)$ staying outside the columns $(i_1,i_2,\dots,i_m)$ in $A_U^{(I)}$. In other words, we have *an affine chart* $\mathbb{A}^{m(d-m)}\xrightarrow{\sim}\mathfrak{U}_I\subset\mathrm{Gr}(m,d)$

which covers an open dense subset of the grassmannian. The charts $\mathfrak{U}_I$ are called *standard* and cover the whole of $\mathrm{Gr}(m,d)$ when I runs through the length m increasing subsets in $\{1,2,\ \dots\ ,d\}$.

Exercise 6.5. Write down the explicit *transition functions* between the standard affine charts $\mathfrak{U}_{12}$ and $\mathfrak{U}_{23}$ on $\mathrm{Gr}(2,4)$.

Exercise 6.6*. If you had deal with smooth topology, check that real and complex grassmannians are the smooth (moreover, analytic) manifolds.

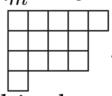
6.9. Cell decomposition. The Gauss method shows that for any $U \subset V$ there exists *a unique* base $\{u_1,u_2,\dots,u_m\}$, of U, such that the corresponding matrix $A_U = (a_{\mu\nu})$ is *a strong step matrix*, that is, $(a_{\mu\nu})$ contains the identity $m\times m$-submatrix, say in columns $(j_1,j_2,\dots,j_m)$, such that each row vanishes at the left of the unity coming from this identity submatrix (i. e. for all $i=1,2,\ \dots\ ,m$ we have $a_{ij}=0 \quad \forall\, j<j_i$).

Exercise 6.7. Prove that different strong step matrices give different subspaces in V.

So, there exist a bijection between $\mathrm{Gr}(m,d)$ and the set of all strong step matrices. The last one splits into disjoint union of the affine spaces. Namely, all strong step matrices that contain the identity submatrix in the fixed columns $I=(i_1,i_2,\dots,i_m)$ have exactly

$$mn-m^2-(i_1-1)-(i_2-2)-\ \cdots\ -(i_m-m)=\dim\mathrm{Gr}(m,d)-\sum_{\nu=1}^{m}(i_\nu-\nu)$$

free entries to put there any numbers from $\Bbbk$. Hence, topologically, $\mathrm{Gr}(m,d)$ is a disjoint union of $\binom{d}{m}$ affine cells $\mathfrak{A}_I$ enumerated by length m increasing subsets $I\subset\{1,2,\ \dots\ ,d\}$. The I-th cell is homeomorphic to the affine space and has codimension $\sum\limits_{\nu=1}^{m}(i_\nu-\nu)$ in $\mathrm{Gr}(m,d)$.

6.10. Young diagram notations. Traditionally, the ν-th difference $(i_\nu-\nu)$ in a length m increasing subset $I\subset\{1,2,\ \dots\ ,d\}$ is denoted by $\lambda_{m+1-\nu}$ in order to have *a partition* $(d-m)\geqslant\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m\geqslant 0$ instead of the increasing collection $1\leqslant i_1<i_2<\ \cdots\ <i_m\leqslant d$. By the definition, a partition λ is a not increasing collection of non negative integers $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)$. A *length* $\ell(\lambda)$ is a number of the last non zero element in λ. A *weight* of λ is $|\lambda|\overset{\text{def}}{=}\sum\limits_\nu\lambda_\nu$. A *Young diagram* of λ is a `flushleft`'ed collection of cell rows whose lengths are $\lambda_1,\lambda_2,\dots,\lambda_m$. For example, the partition $\lambda=(5,4,4,1)$ has length $\ell(\lambda)=4$ weight $|\lambda|=14$ and Young diagram . In other words, the partition is just the Young diagram, its weight is the number of cells, and its length is the number of rows.

Exercise 6.8. Check that there is a bijection between the length m increasing subsets $I\subset\{1,2,\ \dots\ ,d\}$ and the Young diagrams contained in the rectangle of size $m\times(d-m)$.

In terms of Young diagrams, the grassmannian $\mathrm{Gr}(m,d)$ is decomposed into the disjoint union of affine cells enumerated by the Young diagrams contained in the $m\times(d-m)$ - rectangle. λ-th cell has codimension $|\lambda|$ and is isomorphic to $\mathbb{A}^{m(d-m)-|\lambda|}$. In particular there is a unique 1-point cell, which has codimension $m\,(d-m)$ and corresponds to the rectangle itself, and a unique open dense cell of codimension zero, which corresponds to the empty diagram and coincides with the standard affine chart $\mathfrak{U}_{\{1,2,\dots,m\}}$. A topological closure of λ-th affine cell is called *a Schubert cycle* and denoted by σ_λ.

Exercise 6.9. Check that 6 Schubert cycles on the Plücker quadric $\mathrm{Gr}(2,4)\simeq Q_P\subset\mathbb{P}_5$ are: $\sigma_{00}=Q_P$; $\sigma_{22}=p=(0:0:0:0:0:1)\in\mathbb{P}_5$; $\sigma_{10}=Q_P\cap T_pQ_P$; $\sigma_{11}=\pi_\alpha(O)$, where $O=(0:0:0:1)\in\mathbb{P}_3$; $\sigma_{20}=\pi_\beta(\Pi)$, where $\Pi\subset\mathbb{P}_3$ is given by $x_0=0$; $\sigma_{21}=\pi_\alpha(O)\cap\pi_\beta(\Pi)$.

Remark for who studied the topology. Clearly, the Schubert cycles generate the homologies of $\mathrm{Gr}(m,d)$. Moreover, for the complex grassmannian they form the base of $H_*(\mathrm{Gr}(m,\mathbb{C}^d),\mathbb{Z})$ over $\mathbb{Z}$. It is a nice (but not simle) combinatorial problem, to express the (topological) intersections of the Schubert cycles in terms of the Schubert cycles. The corresponding technique is known as *a Schubert calculus* and is described in Griffits-Harris, Fulton-Harris and Macdonald. Roughly speaking, *the homology ring* $H_*(\mathrm{Gr}(m,\mathbb{C}^d),\mathbb{Z})$ is isomorphic to the truncated ring of symmetric polynomials via sending the Schubert cycles σ_λ to the Schur polynomials s_λ.

Exercise 6.10*. Show that $\sigma_{10}\sigma_{21}=\sigma_{20}^2=\sigma_{11}^2=\sigma_{22}$, $\sigma_{20}\sigma_{11}=0$, and $\sigma_{10}^2=\sigma_{20}+\sigma_{11}$ in the integer homology ring of complex $\mathrm{Gr}(2,4)$.

HINT. To calculate σ_{10}^2 realize σ_{10} as $\sigma_{1,0}(\ell) = Q_P \cap T_{\mathrm{u}(\ell)}Q_P = \{\ell'' \subset \mathbb{P}_3 \mid \ell \cap \ell'' \neq \varnothing\}$. Then, taking two *intersecting* lines ℓ and ℓ' in $\mathbb{P}_3$ we get $\sigma_{10}(\ell) \cap \sigma_{10}(\ell') = \pi_\alpha(O) \cup \pi_\beta(\Pi)$, where $O = \ell \cap \ell'$ and Π is spaned by ℓ and ℓ'.

6.11. Plücker equations In general case, an image of the Plücker embedding $\mathrm{Gr}(m, V) \overset{\mathrm{u}}{\hookrightarrow} \mathbb{P}(\Lambda^m V)$, i. e. the variety of decomposable quadratic grassmannian polynomials, is described by the quadratic Plücker relations considered in n° 5.5.2 and generalizing those we written in n° 6.2.2 and n° 6.6.2 for $\dim V = 4$. Note that in the latter particular case we could write four generic relations from n° 5.5.2 that correspond to all possible distributions of $4 = 3 + 1$ indexes $\{1, 2, 3, 4\}$ between K and J.

Exercise 6.11. Check that they all produce the same quadratic equation $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0$ on 2×2 - minors A_{ij} of 2×4 - matrix A.

§7. Working example: Veronese curves.

In this section we always assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk \neq 2$.

7.1. Linear span of Veronese curve. Recall (see n° 5.4.1) that the Veronese curve $C_n \subset \mathbb{P}_n$ is the image of Veronese's map

$$\mathbb{P}_1 = \mathbb{P}(U^*) \xhookrightarrow{v_n} \mathbb{P}_n = \mathbb{P}(S^n U^*) \,,$$

which takes a linear form $\alpha = \alpha_0 t_0 + \alpha_1 t_1 \in U^*$ to its n-th power:

$$v_n(\alpha) = \alpha^n = \sum \binom{n}{i} \alpha_0^i \alpha_1^{n-i}\, t_0^i t_1^{n-i} \,.$$

As in n° 5.4.1, we write polynomials $f \in S^n U^*$ in the form $f = \sum \binom{n}{i} a_i\, t_0^i t_1^{n-i}$ and use a_i as homogeneous coordinates on $\mathbb{P}_n = \mathbb{P}(S^n U^*)$.

Given an arbitrary hyperplane $\pi = \{a \in \mathbb{P}_n \mid \sum A_i a_i = 0\}$, the intersection $C_n \cap \pi$ consists of all $a = a(\alpha)$ that satisfy the equation $\sum\limits_i A_i \alpha_0^i \alpha_1^{n-i} = 0$ whose left side is non zero polynomial of degree n. So, a hyperplane section of Veronese's curve always consists[1] of n points counted with appropriate multiplicities (typically, of n *distinct* points). In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

7.1.1. COROLLARY (ARONHOLD PRINCIPLE). *To prove that some linear in f assertion holds for all polynomials f, it is enough to verify it only for all powers of all linear forms.* □

Exercise 7.1. Use the Aronhold principle to give another proof of the Taylor formula (5-6).

7.2. Projecting twisted cubic. Let us describe all plane projections of the twisted cubic $C_3 \subset \mathbb{P}_3 = \mathbb{P}(S^3 U^*)$. Up to a projective isomorphism, the projection does not depend on the choice of a target plane as soon as the center is fixed, because the projection of one target plane onto another gives an linear isomorphism between the projection images. Let $p = p(t) = \ell_1(t)\ell_2(t)\ell_3(t) \in \mathbb{P}_3 = \mathbb{P}(S^3 V^*)$ be a projection center. After some parameter change we can suppose that either $\ell_1 = \ell_2 = \ell_3 = t_0^2$, (this means that $p \in C$), or $\ell_1 = \ell_2 = t_0$, $\ell_3 = t_1$ [2], or $\ell_1 = (t_0 + t_1)$, $\ell_2 = (t_0 + \omega\, t_1)$, $\ell_3 = (t_0 + \omega^2 t_1)$, where $\omega = \sqrt[3]{1} \neq 1$ (that is, $p(t) = t_0^3 + t_1^3$ has 3 distinct roots).

In the first case $p = (1:0:0:0)$; take a target plane to be $a_0 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_1 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2)$ and coincides with the plane Veronese conic $x_0 x_2 = x_1^2$.

In the second case $p = (0:1:0:0)$; take a target plane to be $a_1 = 0$ with the coordinates $(x_0 : x_1 : x_2) = (a_0 : a_2 : a_3)$. Then the projection is given by parametric equations $(x_0 : x_1 : x_2) = (\alpha_0^3 : \alpha_0\alpha_1^2 : \alpha_1^3)$ and in the affine chart $\{x_0 = 1\}$ it turns into $x = \alpha^2$, $y = \alpha^3$, where $x = x_1/x_0$, $y = x_2/x_0$, and $\alpha = \alpha_1/\alpha_0$. So, we get a curve $y^2 = x^3$ or, in the homogeneous coordinates, $x_2^3 = x_3^2 x_0$. This curve is called *a cuspidal cubic*, because of the singularity form at the origin.

In the third case $p = (1:0:0:1)$; take a target plane $\pi = \{a_2 = 0\}$ with the coordinates $(x_0 : x_1 : x_2) = ((a_0 - a_1) : a_1 : a_2)$ (the first three coordinates w.r.t. the base $\{\, t_0^3 , 3\, t_0^2 t_1 \,,\, 3\, t_0 t_1^2 \,,\, t_0^3 + t_1^3 \,\}$). So, the projection from $p = t_0^3 + t_1^3$ gives the parameterized curve $(x_0 : x_1 : x_2) = \left((\alpha_0^3 - \alpha_1^3) : \alpha_0^2\alpha_1 : \alpha_0\alpha_1^2\right)$. In affine chart $x_0 = 1$ we get, like above, $x = \alpha/(1-\alpha^3)$, $y = \alpha^2/(1-\alpha^3)$ or $xy = x^3 - y^3$. This curve has a self intersection point at the origin and is called *a nodal* cubic.

7.2.1. Example: geometric description of rational curves. A plane curve $C \subset \mathbb{P}_2$ is called *rational* if there are 3 coprime homogeneous polynomials $p_0(t)$, $p_1(t)$, $p_2(t)$ of the same degree in $t = (t_0 : t_1)$ such that a map $\mathbb{P}_1 \xrightarrow{\alpha \mapsto (p_0(\alpha) : p_1(\alpha) : p_2(\alpha))} \mathbb{P}_2$ gives (maybe, after removing some finite sets of points from $\mathbb{P}_1$ and C) a bijection between $\mathbb{P}_1$ and C .

Exercise 7.2. Intersecting C with lines, show that $\deg C = \deg p_i$.

[1] recall that we suppose the ground field to be algebraically closed
[2] geometrically, this means that p lies on a tangent line to C_3

When $\deg p_i = d$, a map $(t_0^d : t_0^{d-1}t_1 : \ldots : t_0 t_1^{d-1} : t_1^d) \longmapsto (p_0(t) : p_1(t) : p_2(t))$ defines a projection of the Veronese curve $C_d \subset \mathbb{P}_d$ into some plane $\mathbb{P}_2 \subset \mathbb{P}_d$. So, we have

7.2.2. CLAIM. *Each rational plane curve of degree d is an appropriate projection of the Veronese curve $C_d \subset \mathbb{P}_d$.* □

7.2.3. COROLLARY. *A smooth plane cubic curve is not rational.*

Proof. Rational cubic curve is a plane projection of the twisted cubic $C_3 \subset \mathbb{P}_3$. But such a projection is either a conic or a singular cubic. □

7.3. Simplices inscribed into the Veronese curve. Let $p_i = \xi_i^n$, where $1 \leqslant i \leqslant n$, $\xi_i \in U^*$, be an arbitrary collection of n distinct points on the Veronese curve $C_n \subset \mathbb{P}_n = \mathbb{P}(S^nU^*)$. For each i consider the pencil of hyperplanes passing through $(n-2)$-dimensional face $(p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_n)$ opposite to p_i in the $(n-1)$ dimensional simplex $(p_1, p_2, \ldots, p_n)$. These n pencils are parameterized uniformly by the points of $\mathbb{P}_1^\times = \mathbb{P}(U)$ as follows. For any $\xi \in \mathbb{P}(U^*)$ denote by $\widehat{\xi} \in \mathbb{P}(U)$ the annihilator[1] $\mathrm{Ann}(\xi)$ and for each i take the product $\tau_i = \widehat{\xi}_1, \cdots, \widehat{\xi}_{i-1}\widehat{\xi}_{i+1}, \cdots, \widehat{\xi}_1 \in S^{n-1}U$. Define a plane $\pi_i(u) \subset \mathbb{P}(S^nU^*)$, which corresponds to $u \in \mathbb{P}(U)$ in i-th pencil, as the annihilator of $u\tau_i \in S^nU$. This means that

$$\pi_i(u) = \left\{ f(t) \in S^nU^* \mid \widetilde{f}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = 0 \right\}$$

where $\widetilde{f}$ is the full polarization of f considered as a symmetric multilinear form on U. In particular, for $f(t) = \zeta^n(t) \in C_n$ we have

$$\widetilde{\zeta^n}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = \zeta(u) \prod_{\nu \neq i} \zeta\left(\widehat{\xi}_\nu\right)$$

So, for any $u \in \mathbb{P}(U)$ the plane $\pi_i(u)$ pass through all $p_\nu = \xi_\nu^n$ with $\nu \neq i$ and through the point $p = \xi^n \in C_n$ whose ξ annihilate u (i. e. such that $u = \widehat{\xi}$). In other words,

$$C_n = \bigcup_u \pi_1(u) \cap \pi_2(u) \cap \cdots \cap \pi_n(u) \tag{7-1}$$

Since $\mathrm{PGL}_2(k)$ acts on $\mathbb{P}(S^nU^*)$ via linear variable a linear isomorphism between projective lines is uniquely defined by the images of any 3 distinct points, we get the following corollary.

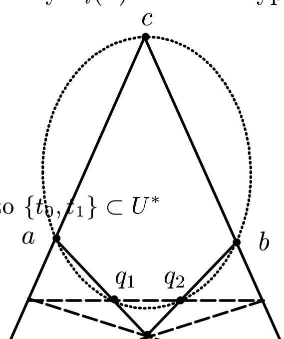
7.3.1. CLAIM. *The Veronese curve is uniquely recovered from any collection of its $(n+3)$ distinct points $a, b, c, p_1, p_2, \ldots, p_n$ as follows. Consider n hyperplane pencils through the $(n-2)$-dimensional faces of the inscribed simplex $(p_1, p_2, \ldots, p_n)$ and parameterize them uniformly by $u \in \mathbb{P}_1$ in such a way that the hyperplanes passing through a, b, c appear in each pencil when $u = 0,\ 1,\ \infty$. Then C_n coincides with the incidence graph* (7-1) *when u runs through the parameter line $\mathbb{P}_1$.* □

7.4. Natural action of $\mathbf{PGL_1 = PGL(U^*)}$ on $\mathbb{P}(S^nU^*)$ induced by the substitutions $(t_0, t_1) \longmapsto (at_0 + bt_1, ct_0 + dt_1)$ sends the Veronese curve to itself. We call it *the reparameterization* of the Veronese curve.

7.4.1. CLAIM. *Let $p_1, p_2, \ldots, p_n, a, b, c \in \mathbb{P}_n = \mathbb{P}(S^nU^*)$ be any $n+3$ points with no $(n+1)$ on the same hyperplane. Then there exists a projective linear isomorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ that sends these points onto Veronese curve C_n; this isomorphism is unique up to a reparameterization of the Veronese curve.*

Proof. For each $i = 1, 2, \ldots, n$ identify $\mathbb{P}_1 = \mathbb{P}(U)$ with a pencil of hyperplanes through $p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_n$ by sending $u = e_0,\ e_1,\ (e_1 - e_0)$ to the hyperplanes that contain a, b, c and denote by $\pi_i(u)$ the u-th hyperplane in the i-pencil. Let the hyperplane $(p_1, p_2, \ldots, p_n)$ appear in i-th pencil when $u = u_i$. We claim that $u_1, u_2, \ldots, u_n \in \mathbb{P}(U)$ are mutually distinct.

Indeed, consider 2-dimensional plane $\Pi = (a, b, c)$ and denote by q_1 and q_2 its intersection with i-th and j-th $(n-2)$-dimensional faces of $(p_1, p_2, \ldots, p_n)$. Sitting in this plane, we will see the picture shown on fig 7◇1. Our i-th and



[1] if $\xi = \alpha_0 t_0 + \alpha_1 t_1$, then $\widehat{\xi} = \alpha_1 e_0 - \alpha_0 e_1$, where $\{e_o, e_1\} \subset U$ is the base of U dual to $\{t_0, t_1\} \subset U^*$

j-th pencils of hyperplanes are represented inside Π by the pencils of lines passing through q_1, q_2 and the hyperplane $(p_1, p_2, \dots, p_n)$ is represented by the line $(q_1 q_2)$ (compare this construction with drawings from the §3).

Exercise 7.3. Show that no 3 of q_1, q_2, a, b, c are collinear (in particular, q_1, q_2 are distinct).

HINT. Use linear generality of $p_1, p_2, \dots, p_n, a, b, c$.

For each pair $i \neq j$ there are two ways to identify the parameter line $\mathbb{P}(U)$ with the pencil of lines passing through $O = (aq_1) \cap (bp_2)$: one takes $u \in U$ to the line through $\pi_i(u) \cap (bc)$, another one takes $u \in U$ to the line through $\pi_j(u) \cap (ac)$. These two parameterizations coincide, because they attach the same u's to a, b, c. Since O, q_1, q_2 are not collinear, two lines corresponding to $u = u_i$, $u = u_j$ (they join O with $\pi_i(u_i) \cap (bc) = (q_1 q_2) \cap (bc)$ and $\pi_j(u_j) \cap (ac) = (q_1 q_2) \cap (ac)$ respectively) are distinct, i. e. $u_i \neq u_j$.

Now, denote by Γ the incidence graph (7-1) build from our current pencils of hyperplanes. Let $u_i = \mathrm{Ann}\,(\xi_i)$ for $\xi_1, \xi_2, \dots, \xi_n \in U^*$. By the n° 1.11.1, there exists a unique projective linear automorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends $p_i \longmapsto \xi_i^n$ for $1 \leqslant i \leqslant n$, $a \longmapsto t_1^n$, and $b \longmapsto t_0^n$. It identifies Γ with the Veronese curve, because it sends the hyperplane pencil through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ to the one through $\xi_1^n, \dots \xi_{i-1}^n, \xi_{i+1}^n, \xi_n^n$ in such a way that $\pi_i(u)$ goes to the hyperplane through ξ^n as soon as $\xi = \mathrm{Ann}\, u$. Indeed, this takes place for $u = e_1$, $u = e_0$ and $u = u_i$ when a, b and p_i go to t_0^n, t_1^n and ξ_i^n. Hence, this holds for each u and for $u = e_1 - e_0$ we have $c \longmapsto (t_0 + t_1)^n$. This proves the existence.

Uniqueness follows from the above construction as well. Namely, after appropriate parameter change we can suppose that the isomorphism in question sends a, b, c to t_0^n, t_1^n and $(t_0 + t_1)^n$. So, it induces the uniform parameterization of hyperplane pencils trough p_ν and this parameterization coincides with the above one. So, the images of $p_1, p_2, \dots, p_n$ are uniquely recovered as $\xi_i = \mathrm{Ann}\,(u_i)$. $\square$

§8. Commutative algebra draught.

8.1. Noetherian rings. We write $(f_1, f_2, \ldots, f_m)$ for the ideal $\{g_1 f_1 + g_2 f_2 + \cdots + g_m f_m \mid g_\nu \in A\}$ spanned (as A-module) by $\{f_1, f_2, \ldots, f_m\} \subset A$. A commutative ring A is called *Noetherian*, if it satisfies the next lemma:

8.1.1. LEMMA. *The following properties of a commutative ring A are mutually equivalent*

(1) *any collection of elements $\{f_\nu\}$ contains a finite subset generating the same ideal as the whole set;*

(2) *any ideal admits a finite set of generators;*

(3) *for any infinite chain of embedded ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ there exists $n \in \mathbb{N}$ such that $I_\nu = I_n \quad \forall \nu \geqslant n$.*

Proof. Clearly, (1) $\Rightarrow$ (2). To deduce (3) from (2), take a finite set of generators for the ideal $\bigcup I_\nu$; since they all belong to some I_n, we get $I_\nu = I_n$ for $\nu \geqslant n$. Finally, (1) follows from (3) applied to the chain $I_n = (f_1, f_2, \ldots, f_n)$, where f_i are chosen from $\{f_\nu\}$ in order to have $f_\nu \notin (f_1, f_2, \ldots, f_{\nu-1})$. □

8.1.2. THEOREM (HILBERT'S THEOREM ON A BASIS). *If A is Noetherian, then $A[x]$ is Noetherian.*

Proof. Let $I \subset A[x]$ be an ideal. We write $L_d \subset A$ for a set of leading coefficients of all degree d polynomials in I. Clearly, each L_d and $L_\infty \overset{\text{def}}{=} \bigcup_d L_d$ are ideals in A. Let L_∞ be generated by $a_1, a_2, \ldots, a_s \in A$ coming from $f_1^{(\infty)}, f_2^{(\infty)}, \ldots, f_{s_\infty}^{(\infty)} \in I$ and let $\max_\nu(\deg f_\nu) = m$. Similarly, write $f_1^{(k)}, f_2^{(k)}, \ldots, f_{s_k}^{(k)}$ for the polynomials whose leading coefficients span the ideal L_k for $0 \leqslant k \leqslant m-1$. It is easy to see that I is spanned by $s_0 + \cdots + s_{m-1} + s_\infty$ polynomials $f_\nu^{(\mu)}$. □

Exercise 8.1. Verify the latter claim neatly.

8.1.3. COROLLARY. *If A is Noetherian, then $A[x_1, x_2, \ldots, x_n]$ is Noetherian.* □

8.1.4. COROLLARY. *Any finitely generated $\Bbbk$-algebra is Noetherian for any field $\Bbbk$.*

Proof. A polynomial algebra $\Bbbk[x_1, x_2, \ldots, x_n]$ is Noetherian by the previous corollary. Any its factor algebra A is Noetherian as well: full preimage of any ideal $I \subset A$ under the factorizing morphism $\Bbbk[x_1, x_2, \ldots, x_n] \twoheadrightarrow A$ is an ideal in $\Bbbk[x_1, x_2, \ldots, x_n]$, i.e. admits a finite set of generators, whose classes span I over A, certainly. □

8.2. Integrality. Let $A \subset B$ be two commutative rings. An element $b \in B$ is called *integer* over A, if it satisfies the conditions from n° 8.2.1 below. If all $b \in B$ are integer over A, then B is called an *integer extension* of A or an *integer A-algebra*.

8.2.1. LEMMA. *The following properties of an element $b \in B \supset A$ are pairwise equivalent:*

(1) $b^m = a_1 b^{m-1} + \cdots + a_{m-1} b + a_0$ *for some $m \in \mathbb{N}$ and some $a_1, a_2, \ldots, a_m \in A$;*

(2) *A-module spanned by all nonnegative powers $\{b^i\}_{i \geqslant 0}$ admits a finite set of generators;*

(3) *there exist a finitely generated faithful*[1] *A-submodule $M \subset B$ such that $bM \subset M$.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are trivial. To deduce (1) from (3), let $\{e_1, e_2, \ldots, e_m\}$ generate M over A and let the multiplication map $M \xrightarrow{m \mapsto bm} M$ be presented by a matrix Y, i.e.

$$(be_1, be_2, \ldots, be_m) = (e_1, e_2, \ldots, e_m) \cdot Y .$$

Note that if A-linear map $M \xrightarrow{\varkappa} M$ takes $(e_1, e_2, \ldots, e_m) \longmapsto (e_1, e_2, \ldots, e_m) \cdot X$, where X is a square matrix with entries in A, then the Sylvester relation $\det X \cdot \mathrm{Id} = \widehat{X} \cdot X$ implies an inclusion $(\det X) \cdot M \subset \varkappa(M)$. In our case this can be applied to the zero operator $M \xrightarrow{\varkappa} 0$ and the matrix $X = b \cdot \mathrm{Id} - Y$. We conclude that

[1] A-module M is called *faithful*, if $aM = 0$ implies $a = 0$ for $a \in A$

the multiplication by $\det(b\cdot \mathrm{Id}-Y)$ annihilates M. Since M is faithful, $\det(b\cdot\mathrm{Id}-Y)=0$. This is a polynomial equation on b with the coefficients in A and the leading term b^n as required in (1). $\square$

8.2.2. Example: integer algebraic numbers. Let $K\supset\mathbb{Q}$ be a finite dimensional[1] field extension; then elements $z\in K$ are called *algebraic numbers*. Such a number z is integer over $\mathbb{Z}$ iff there are some $\vartheta_1,\vartheta_2,\dots,\vartheta_m\in K$ such that the multiplication by z sends their $\mathbb{Q}$-linear span to itself and is presented there by a matrix whose entries belong to $\mathbb{Z}$.

8.2.3. Example: invariants of a finite group action. Let a finite group $\mathfrak{G}$ act on a $\Bbbk$-algebra B via $\Bbbk$-algebra automorphisms $B\xrightarrow{g}B$, $g\in\mathfrak{G}$, and let $A=B^{\mathfrak{G}}=\{a\in B\mid ga=a\ \forall g\in\mathfrak{G}\}$ be the subalgebra of $\mathfrak{G}$-invariants. Then B is an integer extension of A. Indeed, if $b_1,b_2,\dots,b_s\in B$ form a $\mathfrak{G}$-orbit of any given $b=b_1\in B$, then the polynomial $\beta(t)=\prod(t-b_i)$ is monic[2], lies in $A[t]$, and annihilates b.

8.3. Integer closures. A set of all $b\in B$ that are integer over a subring $A\subset B$ is called *an integer closure* of A in B. If this closure coincides with A, then A is called *integrally closed* in B.

8.3.1. LEMMA. *The integer closure of A is a subring in B (in particular, ab is integer for any $a\in A$ as soon b is integer). If $C\supset B$ is an other commutative ring and $c\in C$ is integer over an integer closure of A in B, then c is integer over A as well (in particular, any integer B-algebra is an integer A-algebra as soon B is an integer A-algebra).*

Proof. If $p^m=x_{m-1}p^{m-1}+\dots+x_1p+x_0$, $q^n=y_{n-1}q^{n-1}+\dots+y_1q+y_0$ for $p,q\in B$, $x_\nu,y_\mu\in A$, then A-module spanned by p^iq^j with $0\leqslant i\leqslant(m-1)$, $0\leqslant j\leqslant(n-1)$ is faithful (it contains 1) and goes to itself under the multiplication by both $p+q$ and pq. Similarly, if $c^r=z_{r-1}c^{r-1}+\dots+z_1c+z_0$ and all z_ν are integer over A, then a multiplication by c preserves a faithful A-module spanned by a sufficient number of products $c^iz_1^{j_1}z_2^{j_2}\cdots z_r^{j_r}$. $\square$

8.3.2. COROLLARY (GAUSS LEMMA). *For any two commutative rings $A\subset B$ let $f(x),g(x)\in B[x]$ be two monic polynomials. Then all coefficients of $h(x)=f(x)g(x)$ are integer over A iff all coefficients of both $f(x)$, $g(x)$ are integer over A.*

Proof. There exists[3] a ring $C\supset B$ such that $f(x)=\prod(t-\alpha_\nu)$ and $g(x)=\prod(t-\beta_\mu)$ in $C[x]$ for some $\alpha_\nu,\beta_\mu\in C$. By n° 8.3.1, all coefficients of $h(x)=\prod(t-\alpha_\nu)\prod(t-\beta_\mu)$ are integer over $A\iff$ all α_ν,β_μ are integer over $A\iff$ all coefficients of $f(x)$ and $g(x)$ are integer over A. $\square$

8.3.3. LEMMA. *Let $B\supset A$ be integer over A. If B is a field, then A is a field. Vice versa, if A is a field and B has no zero divisors, then B is a field.*

Proof. If B is a field integer over A, then any non zero $a\in A$ has an inverse $a^{-1}\in B$, which satisfy an equation $a^{-m}=\alpha_1a^{1-m}+\dots+\alpha_{m-1}a^{-1}+\alpha_0$ with $\alpha_\nu\in A$. We multiply the both sides by a^{m-1} and get

$$a^{-1}=\alpha_1+\dots+\alpha_{m-1}a^{m-2}+\alpha_0a^{m-1}\in A\,.$$

Conversely, if A is a field and B is an integer A-algebra, then all non negative integer powers b^i of any $b\in B$ form a finite dimensional vector space V over A. If $b\neq0$ and there are no zero divisors in B, then $x\longmapsto bx$ is an injective linear operator on V, i.e. an isomorphism. A preimage of $1\in V$ is b^{-1}. $\square$

8.3.4. Example: algebraic elements and minimal polynomials. If $A=\Bbbk$ is a field and $B\supset\Bbbk$ is a $\Bbbk$-algebra, then $b\in B$ is integer over $\Bbbk$ iff b satisfies $f(b)=0$ for some $f\in\Bbbk[x]$. Traditionally, such b is called *algebraic* over $\Bbbk$ rather than *integer*.

We write $\Bbbk[b]$ for a $\Bbbk$-linear span of nonnegative integer powers $\{b^n\}_{n\geqslant0}$. If $\exists\,b^{-1}\in B$, then we write $\Bbbk(b)$ for a $\Bbbk$-linear span of all integer powers $\{b^n\}_{n\in\mathbb{Z}}$. Clearly, $\Bbbk[b]\subset B$ is the minimal $\Bbbk$-subalgebra containing 1 and b. In other terms, $\Bbbk[b]=\mathrm{im}\,(\mathrm{ev}_b)=\Bbbk[x]/\ker(\mathrm{ev}_b)$, where $\mathrm{ev}_b:\Bbbk[x]\xrightarrow{f(x)\mapsto f(b)}B$ is *an evaluation homomorphism*.

If b is algebraic, then $\ker(\mathrm{ev}_b)=(f)$ for some non zero $f\in\Bbbk[x]$, because $\Bbbk[x]$ is a principal ideal domain. This f is fixed uniquely as a monic polynomial of lowest degree such that $f(b)=0$; it is called *the minimal polynomial*

[1] as a vector space over $\mathbb{Q}$

[2] a polynomial is called *monic* or *unitary*, if its leading coefficient equals 1

[3] For any commutative ring A and any monic non constant $f(x)\in A[x]$ there exists a commutative ring $C\supset A$ such that $f(x)=\prod(x-c_\nu)$ in $C[x]$ for some $c_\nu\in C$. It is constructed inductively as follows. Consider a factor ring $B=A[x]/(f)$ (which contains A as the congruence classes of constants) and put $b\overset{\text{def}}{=}x\pmod f\in B$. Then $f(b)=0$ in $B[x]$. Hence the residue after dividing $f(x)$ by $(x-b)$ in $B[x]$ vanishes and we get the factorization $f(x)=(x-b)h(x)$ with $h(x)\in B[x]$. Now repeat the procedure for h, B instead of f, A e.t.c.

of b over $\Bbbk$. Note that in this case $1,\ b,\ b^2,\ \dots,\ b^{\deg(f)-1}$ form a basis for the vector space $\Bbbk[b]$ over $\Bbbk$ and if B has no zero divisors, then $\Bbbk[b]$ is a field by n° 8.3.3 (in particular, the minimal polynomial of b has to be irreducible).

If b is not algebraic, then $\ker(\mathrm{ev}_b) = 0$ and $\Bbbk[b] \simeq \Bbbk[x]$ is a polynomial ring. It is infinite dimensional as a vector space over $\Bbbk$ and it is not a field.

We generalize this alternative in n° 8.5.1 below.

8.3.5. LEMMA. *Let $K = Q(A)$ be a fraction field of a commutative ring A without zero divisors, B be any K-algebra, and $b \in B$ be algebraic over K with minimal polynomial $f \in K[x]$. If b is integer over A, then all coefficients of f are integer over A.*

Proof. Since b is integer, $g(q) = 0$ for some monic $g \in A[x]$. Then $g = fh$ in $K[x]$ for some monic $h \in K[x]$ and all the coefficients of g, h are integer over A by the Gauss lemma from n° 8.3.2. □

8.4. Normal rings.

A commutative ring A without zero divisors is called *normal*, if it is integrally closed in $Q(A)$. Certainly, any field is normal.

Exercise 8.2. Show that the ring of integer numbers $\mathbb{Z}$ is normal.

HINT. A polynomial $a_0t^m + a_1t^{m-1} + \dots + a_{m-1}t + a_m \in \mathbb{Z}[t]$ annihilates a fraction $p/q \in \mathbb{Q}$ with coprime $p, q \in \mathbb{Z}$ only if $q|a_0$ and $p|a_m$

8.4.1. COROLLARY. *Let A be a normal ring with the fraction field $K = Q(A)$. If $f \in A[x]$ is factorized in $K[x]$ as $f = gh$, where both g, h are monic, then $g, h \in A[x]$.*

Proof. Indeed, all the coefficients of g, h are integer over A by n° 8.3.2. □

8.4.2. COROLLARY. *Let A be normal ring with the fraction field $K = Q(A)$ and B be any K-algebra. Then $b \in B$ is integer over A iff it is algebraic over K and its minimal (over K) polynomial lies in $A[x]$.*

Proof. This follows immediately from n° 8.3.5. □

8.5. Finitely generated commutative $\Bbbk$-algebras.

Let $\Bbbk$ be an arbitrary field. A commutative $\Bbbk$-algebra B is called *finitely generated*, if there is a $\Bbbk$-algebra epimorphism $k[x_1, x_2, \dots, x_m] \xrightarrow{\pi} B$. In this case the images $b_i = \pi(x_i) \in B$ are called *algebra generators* for B over $\Bbbk$.

8.5.1. LEMMA. *A finitely generated $\Bbbk$-algebra B can be a field only if each $b \in B$ is algebraic over $\Bbbk$.*

Proof. Let B be a field and $\{b_1, b_2, \dots, b_m\}$ be some algebra generators for B over $\Bbbk$. We use induction over m. The case $m = 1$, $B = \Bbbk[b]$ was considered in n° 8.3.4. For $m > 1$, if b_m is algebraic over $\Bbbk$, then $\Bbbk[b_m]$ is a field and B is algebraic over $\Bbbk[b_m]$ by the inductive assumption. Hence, by n° 8.3.1, B is algebraic over $\Bbbk$ as well. So, it is enough to show that b_m *must* be algebraic over $\Bbbk$ as soon $m > 1$.

Suppose the contrary: let b_m be not algebraic. Then $\Bbbk(b_m)$ is isomorphic to the field $\Bbbk(x)$, of rational functions in one variable, via sending $b_m \longmapsto x$. By the inductive assumption, B is algebraic over $\Bbbk(b_m)$ and $b_1, b_2, \dots, b_{m-1}$ satisfy polynomial equations with coefficients in $\Bbbk(b_m)$. Multiplying these equations by appropriate polynomials in b_m, we can put their coefficients into $\Bbbk[b_m]$ and make all their leading coefficients to be equal to the same polynomial, which we denote by $p(b_m) \in \Bbbk[b_m]$.

Now, B is integer over a subalgebra $F \subset B$ generated over $\Bbbk$ by b_m and $q = 1/p(b_m)$. By n° 8.3.3, F is a field. So, there exists a polynomial $g \in \Bbbk[x_1, x_2]$ such that $g(b_m, q)$ is inverse to $1 + q$ in F. Let us write the rational function $g(x\,,\, 1/p(x)) \in \Bbbk(x)$ as $h(x)/p^k(x)$, where $h \in \Bbbk[x]$ is coprime to $p \in \Bbbk[x]$. Multiplying the both sides of

$$\left(1 + \frac{1}{p(b_m)}\right) \frac{h(b_m)}{p^k(b_m)} = 1$$

by $p^{k+1}(b_m)$, we get for b_m a polynomial equation $h(b_m)\,(p(b_m) + 1) = p^{k+1}(b_m)$. It is nontrivial, because $h(x)(1 + p(x))$ is not divisible by $p(x)$. Hence, b_m should be algebraic over $\Bbbk$. □

8.6. Hilbert's Nullstellensatz.

Let us write $V(I) = \{a \in \mathbb{A}_n \mid f(a) = 0 \quad \forall\, f \in I\ \} \subset \mathbb{A}^n$ for affine algebraic variety defined by a system of polynomial equations $I \subset \Bbbk[x_1, x_2, \dots, x_n]$. Certainly, $V(I)$ is not changed when I is extended to an ideal spanned by I.

Vice versa, for any subset $V \subset \mathbb{A}^n$ we write $I(V) = \{f \in k[x_1, x_2, \dots, x_n] \mid f|_V \equiv 0\}$ for a set of all polynomials vanishing along V. Clearly, $I(V)$ is always an ideal and $I(V(I)) \supset I$ for any ideal I. Generically, the latter inclusion is proper. For example, if $I = (x^2) \in \mathbb{C}[x]$, then $V(I) = \{0\} \subset \mathbb{A}^1(\mathbb{C})$ and $I(V(I)) = (x)$.

8.6.1. THEOREM (WEEK NULLSTELLENSATZ). *Let* $\Bbbk$ *be an arbitrary algebraically closed field and* $I \subset \Bbbk[x_1, x_2, \dots, x_n]$ *be an ideal. Then* $V(I) = \varnothing$ *iff* $1 \in I$.

Proof. If $1 \in I$, then $V(I) = \varnothing$, certainly. Let $I \subset \Bbbk[x_1, x_2, \dots, x_n]$ be a proper ideal. We must find a point $p \in \mathbb{A}_n$ such that $f(p) = 0$ for all $f \in I$. We can assume that I is *maximal*, i.e. any $g \notin I$ is invertible modulo I. Indeed, otherwise an ideal J generated by g and I would be proper and strictly larger than I and we could replace I by J; a finite chain of such replacements leads to some maximal ideal.

As soon I is maximal the factor algebra $K = \Bbbk[x_1, x_2, \dots, x_n]/I$ is a field. Hence, any element of K is algebraic over $\Bbbk \subset K$ by n° 8.5.1. Since $\Bbbk$ is algebraically closed, this means that any polynomial is (mod I)-congruent to some constant. Let $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ be the constants presenting basic linear forms $x_1, x_2, \dots, x_n$(mod I). Then any polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ is (mod I) congruent to $f(\vartheta_1, \vartheta_2, \dots, \vartheta_n) \in \Bbbk$. In particular, $f(\vartheta_1, \vartheta_2, \dots, \vartheta_n) = 0$ for any $f \in I$ as required. □

8.6.2. COROLLARY (STRONG NULLSTELLENSATZ). *Let* $\Bbbk$ *be an arbitrary algebraically closed field and* $I \subset \Bbbk[x_1, x_2, \dots, x_n]$ *be an ideal. Then* $f \in I(V(I))$ *iff* $f^k \in I$ *for some* $k \in \mathbb{N}$.

Proof. If $V(I) = \varnothing$, there is nothing to prove. Clearly, vanishing of f^k along $V(I)$ always implies vanishing of f itself. So, the theorem is reduced to the following statement: if f vanishes along a *nonempty* algebraic variety $V(I)$, then $f^k \in I$ for some k.

To prove it, consider bigger affine space $\mathbb{A}^{n+1}$ with coordinates $(t, x_1, x_2, \dots, x_n)$ and identify the initial $\mathbb{A}^n$ with the hyperplane $t = 0$ in this bigger space. If $f \in \Bbbk[x_1, x_2, \dots, x_n] \subset \Bbbk[t, x_1, x_2, \dots, x_n]$ vanishes along $V(I)$, then an ideal $J \subset \Bbbk[t, x_1, x_2, \dots, x_n]$ spanned by I and a polynomial $g(t, x) = 1 - t\,f(x)$ has empty zero set $V(J) \subset \mathbb{A}^{n+1}$, because $g(x, t) \equiv 1$ on $V(I)$. By the week Nullstellensatz $1 \in J$, i.e.

$$q_0(x,t)(1 - tf(x)) + q_1(t,x)f_1(x) + \dots + q_s(x,t)f_s(x) = 1 \tag{8-1}$$

for appropriate $q_0, q_1, \dots, q_s \subset \Bbbk[t, x_1, x_2, \dots, x_n]$, $f_1, f_2, \dots, f_s \subset I$. Consider a homomorphism

$$\Bbbk[t, x_1, x_2, \dots, x_n] \longrightarrow \Bbbk(x_1, x_2, \dots, x_n)$$

that sends $t \longmapsto 1/f(x)$, $x_\nu \longmapsto x_\nu$. It takes (8-1) to the identity

$$q_1(1/f(x)\,,\, x)\, f_1(x) + \dots + q_s(1/f(x)\,,\, x)\, f_s(x) = 1$$

inside $\Bbbk(x_1, x_2, \dots, x_n)$. Since I is proper, some of $q_\nu(1/f(x), x)$ actually have nontrivial denominators of the form f^ν. Hence, multiplying by appropriate power f^k, we get an expression $\widetilde{q}_1(x)\, f_1(x) + \dots + \widetilde{q}_s(x)\, f_s(x) = f^k(x)$ with $\widetilde{q}_\nu \in \Bbbk[x_1, x_2, \dots, x_n]$. □

8.7. Factorization. Let A be a commutative ring without zero divisors. An element $q \in A$ is called *irreducible*, if it is not invertible and $q = rs$ implies that one of r, s is invertible. An element $p \in A$ is called *prime*, if it generates a prime ideal in A, i.e. if $A/(p)$ is not zero and has no zero divisors.

Exercise 8.3. Check that p is prime iff it is not invertible and $p|rs$ implies that p divides at least one of r, s.

Exercise 8.4. Show that each prime element is irreducible.

A ring A is called *factorial*, if any $a \in A$ is a finite product of irreducible elements:

$$a = q_1\, q_2\, \cdots\, q_m$$

and such *irreducible factorization* is unique up to multiplication of its factors by invertible elements[1], i.e. given two irreducible factorizations

$$q_1\, q_2\, \cdots\, q_m = a = q_1'\, q_2'\, \cdots\, q_n'\ ,$$

then $m = n$ and (after appropriate renumbering) $q_i = s_i\, q_i'$ for some invertible $s_i \in A$.

8.7.1. LEMMA. *Any factorial ring* A *is normal.*

Proof. Let $\alpha/\beta \in Q(A)$ satisfy a polynomial equation $t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n = 0$, where $\alpha_i \in A$. Then α^n is divisible by β. Since A is factorial, each irreducible divisor of β divides α, that is, $\alpha/\beta \in A$. □

[1] recall that if $a = bs$ for an invertible s, then a and b are called *associated* (with each other) elements of ring

8.7.2. LEMMA. *A Noetherian ring without zero divisors is factorial iff all its irreducible elements are prime.*

Proof. In a Noetherian ring, any element f is a finite product of irreducible elements: in the contrary case f can written as $f = f_1 g_1$, where f_1 is reducible and can be written as $f = f_2 g_2$ an so on *infinitely* many times producing an infinite chain of strictly increasing ideals $(f) \subset (f_1) \subset (f_2) \subset (f_2) \subset \ldots$. Further, if there are no zero divisors and all irreducible elements are prime, then two irreducible factorizations $\prod q_i = \prod q'_j$ have the same number of factors and satisfy $q_i = s_i q'_i$ for some invertible s_i (after appropriate renumbering). Indeed, since prime q'_1 divides $\prod q_i$ it divides some q_i, say q_1. So, $q_1 = s_1 q'_1$, where s_1 is invertible, because q_1 is irreducible. Now we have $q'_1 \left(s_1 \prod\limits_{i \geqslant 2} q_i - \prod\limits_{j \geqslant 2} q'_j \right) = 0$, which implies $s_1 \prod\limits_{i \geqslant 2} q_i = \prod\limits_{j \geqslant 2} q'_j$, and we can replace q_2 by $s_1 q_2$ and use induction over the number of factors.

It remains to note that in factorial ring all irreducible elements are prime: if $ab = pq$, where q is irreducible, then irreducible factorization of either a or b should contain an element sq with invertible s. □

8.7.3. Greatest common divisor. Let A be a factorial ring and $a_1, a_2 \in A$ have the prime factorizations:

$$a_1 = q_1 \cdots q_s q'_{s+1} \cdots q'_m \ , \quad a_2 = q_1 \cdots q_s q'_{s+1} \cdots q'_n \ ,$$

where no q'_i, q'_j are associated (the case $s = 0$, without any q's, is also possible). The product $q_1 \cdots q_s$ (or 1, if $s = 0$) is called *the greatest common divisor* of a_1, a_2 and denoted by $\gcd(a_1, a_2)$. Note that $\gcd(a_1, a_2)$ is defined up to invertible factor. Inductively,

$$\gcd(a_1, a_2, \ldots, a_n) = \gcd\left(\gcd(a_1, a_2, \ldots, a_{n-1}), a_n\right) \ .$$

Given a polynomial $f = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \in A[x]$, then $\gcd(a_1, a_2, \ldots, a_n)$ is called *a content* of f and is denoted by $\mathrm{cont}(f)$.

8.7.4. LEMMA. $\mathrm{cont}(fg) = \mathrm{cont}(f) \cdot \mathrm{cont}(g)$ *for any* $f, g \in A[x]$.

Proof. It is enough to check that $\mathrm{cont}(fg) = 1$, if $\mathrm{cont}(f) = \mathrm{cont}(g) = 1$. If all the coefficients of fg are divisible by some prime $p \in A$, then $fg \pmod p = 0$ in the ring $(A/pA)[x]$, which has no zero divisors, because p is prime. So, either $f \pmod p = 0$ or $g \pmod p = 0$. □

8.7.5. LEMMA. *If A is factorial, then $A[x]$ is factorial as well.*

Proof. Let $\Bbbk = Q(A)$ be the quotient field. By n° 8.7.2, it is enough to show that any irreducible $f \in A[x]$ remains to be irreducible inside the factorial ring $\Bbbk[x]$. Let $f = gh$ in $\Bbbk[x]$. We can write $g(x) = a^{-1} g'(x)$, $h(x) = b^{-1} h'(x)$ for some $a, b \in A$ and $g', h' \in A[x]$ such that $\mathrm{cont}(g') = \mathrm{cont}(h') = 1$. Now $ab\, f = g'h'$, where $\mathrm{cont}(g'h') = 1$ by n° 8.7.4 and $\mathrm{cont}(f) = 1$, because f is irreducible in $A[x]$. Hence. ab is invertible and $h'' = (ab)^{-1} h' \in A[x]$. This leads to the decomposition $f = h'' g'$ inside $A[x]$. □

8.7.6. COROLLARY. *If A is factorial, then $A[x_1, x_2, \ldots, x_n]$ is factorial (in particular, normal).* □

Exercise 8.5. Let $\Bbbk$ be an algebraically closed field of any characteristic, $X \subset \mathbb{A}_n(k)$ be an algebraic hypersurface given by a polynomial equation $f(x_1, x_2, \ldots, x_n) = 0$, where $f \in \Bbbk[x_1, x_2, \ldots, x_n]$, and let $g(x) \in \Bbbk[x_1, x_2, \ldots, x_n]$ vanish at any point of X. Show that g is divisible by any irreducible factor of f

HINT. Since $\Bbbk[x_1, x_2, \ldots, x_n]$ is factorial, the result follows from Hilbert's Nullstellensatz

8.8. Resultant systems. We fix a collection of m degrees $d_1, d_2, \ldots, d_m$ and write $\mathscr{S}_d = \mathbb{P}(S^d V^*)$ for the space of hypersurfaces of degree d in $\mathbb{P}_n = \mathbb{P}(V)$. Let $\mathscr{R} \subset \mathscr{S}_{d_1} \times \mathscr{S}_{d_2} \times \cdots \times \mathscr{S}_{d_m}$ be a set of all hypersurface collections $S_1, S_2, \ldots, S_m \subset \mathbb{P}_n$ such that $\bigcap S_\nu \neq \varnothing$. Then $\mathscr{R}$ is an *algebraic variety*, i. e. can be described by a finite system of multi-homogeneous polynomial equations on the coefficients of forms $(f_1, f_2, \ldots, f_m) \in S^{d_1} V^* \times \cdots \times S^{d_n} V^*$ defining the hypersurfaces $S_1, S_2, \ldots, S_m$. This equation system depends only on n,$d_1, d_2, \ldots, d_m$ and is called *a resultant system*. Indeed, consider an ideal $I \subset \Bbbk[x_0, x_1, \ldots, x_n]$ generated by f_ν. Then $\bigcap S_\nu \subset \mathbb{P}(V)$ is empty $\iff V(I) \subset \mathbb{A}(V)$ either is empty or coincides with the origin $O \in \mathbb{A}(V)$. In the both cases each x_i vanishes along $V(I)$, i. e. by Hilbert's Nullstellensatz $x_i^m \in I$ for some m, that is $S^d V^* \subset I \ \forall d \gg 0$. Since $V(x_0^m, x_1^m, \ldots, x_n^m) = \{O\}$ this condition is also sufficient. So, $\bigcap S_\nu = \varnothing$ iff $\Bbbk$-linear map:

$$\mu_d \ : \ S^{d-d_0} V^* \oplus S^{d-d_1} V^* \oplus \cdots \oplus S^{d-d_n} V^* \xrightarrow{(g_0, g_1, \ldots, g_n) \mapsto \sum g_\nu f_\nu} S^d \qquad (8\text{-}2)$$

is non surjective $\forall\, d \gg 0$. In terms of the standard monomial bases, μ_d is presented by a matrix whose entries are linear forms in the coefficients of f_ν. Since for $d \gg 0$ the dimension of the left side in (8-2) becames greater then the right one[1], $\mathscr{R}$ coincides with the zero set of all $d \times d$ - minors of all μ_d with d large enough. By Hilbert's theorem on a basis, this infinite equation system is equivalent to some finite subsystem. Say, we will see in n° 8.8.2 that for $n = 1$, $m = 2$ the smallest d producing a non trivial restriction is $d = d_0 + d_1 - 1$, when μ_d becomes a square matrix; in this case $\mathscr{R}$ is a hypersurface given by equation $\det \mu_{d_0+d_1-1} = 0$.

8.8.1. Example: projection $\mathbb{P}_m \times \mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^n$ sends algebraic varieties to algebraic varieties, i. e. if

$$X = \{\, (q, p) \in \mathbb{P}_m \times \mathbb{A}^n \mid f_\nu(q, p) = 0 \,\}$$

is given by some polynomial equations $f_\nu(t, x) = 0$ (homogeneous in $t = (t_1, t_2, \dots, t_m) \in \mathbb{P}_m$), then its projection onto $\mathbb{A}^n$ also can be described by a system of polynomial equations.

Indeed, consider f_ν as homogeneous polynomials in t with the coefficients in $\Bbbk[x_1, x_2, \dots, x_n]$. Then the image $\pi(X) \subset \mathbb{A}^n$ consists of all p such that the homogeneous in t polynomials $f_\nu(t, p)$ have a common zero on $\mathbb{P}_m$. As we have seen, this means that their coefficients, which are polynomials in p, satisfy the system of resultant equations.

8.8.2. Example: resultant of two binary forms. If $\Bbbk$ is algebraically closed, then each polynomial $f(t) = a_0\, t^m + a_1\, t^{m-1} + \cdots + a_{m-1}\, t + a_m$ can be factorized as $f(t) = a_0 \prod (t - \vartheta_\nu) = a_m \prod (1 - \vartheta_\nu^{-1} t)$, where $\vartheta_1, \vartheta_2, \dots, \vartheta_m$ are all its roots. In homogeneous world, each degree d homogeneous polynomial

$$A(t_0, t_1) = a_0\, t_1^d + a_1\, t_0\, t_1^{d-1} + a_2\, t_0^2\, t_1^{d-2} + \cdots + a_{d-1}\, t_0^{d-1}\, t_1 + a_d\, t_0^d$$

has similar decomposition $A(t_0, t_1) = \prod\limits_{i=0}^{d} (\alpha_i'' t_0 - \alpha_i' t_1) = \prod\limits_{i=0}^{d} \det \begin{pmatrix} t_0 & t_1 \\ \alpha_i' & \alpha_i'' \end{pmatrix}$, which means that A vanishes at d points $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{P}_1$ with homogeneous coordinates $\alpha_i = (\alpha_i' : \alpha_i')$. In particular, each coefficient a_i, of $A(t)$, is expressed as bihomogeneous degree $(i, d-i)$ polynomial in (α', α''):

$$a_i = (-1)^{d-i} \sigma_i(\alpha', \alpha'') \;, \quad \text{where} \quad \sigma_i(\alpha', \alpha'') = \sum_{\#I = i} \Big(\prod_{i \in I} \alpha_i' \cdot \prod_{j \notin I} \alpha_j'' \Big)$$

(here I runs through all increasing length i subsets in $\{1, 2, \dots, d\}$ and σ_i is a *bi*homogeneous version of the i-th elementary symmetric function).

Now, let us fix two degrees $m, n \in \mathbb{N}$ and consider a polynomial ring $\Bbbk[\alpha', \alpha'', \beta', \beta'']$ in four collections of variables $\alpha' = (\alpha_1', \alpha_2', \dots, \alpha_n')$, $\alpha'' = (\alpha_1'', \alpha_2'', \dots, \alpha_n'')$, $\beta' = (\beta_1', \beta_2', \dots, \beta_m')$, $\beta'' = (\beta_1'', \beta_2'', \dots, \beta_m'')$ Then the product

$$R_{AB} \overset{\text{def}}{=} \prod_{i,j} \left(\alpha_i' \beta_j'' - \alpha_i'' \beta_j' \right) = \prod_{j=1}^{n} A(\beta_j) = (-1)^{mn} \prod_{i=1}^{m} B(\alpha_i) \;,$$

vanishes iff two homogeneous binary forms $A(t_0, t_1) = \sum\limits_{i=0}^{n} a_i\, t_0^i\, t_1^{n-i}$ and $B(t_0, t_1) = \sum\limits_{j=0}^{m} b_j\, t_0^j\, t_1^{m-j}$ (whose coefficients $a_i = (-1)^{n-i} \sigma_i(\alpha', \alpha'')$, $b_j = (-1)^{m-j} \sigma_j(\beta', \beta'')$ have a common root. Clearly, $R_{A,B}$ is bihomogeneous of bidegree (mn, mn) in (α, β) and may be expressed in terms of the coefficients of A, B. This expression is called *a resultant* of the polynomials $A(t_0, t_1)$, $B(t_0, t_1)$ and generates the ideal of all resultant relations for two binary forms. More precisely, $R_{A.B}$ up to a scalar factor coincides with *the Silvester determinant*

$$\det \underbrace{\begin{pmatrix} a_0 & a_1 & \dots & a_n & & & \\ & a_0 & a_1 & \dots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & & & \\ & b_0 & b_1 & \dots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \dots & b_m \end{pmatrix}}_{m+n} \begin{matrix} \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} n \end{matrix}$$

Indeed, consider a vector space U with a basis $\{t_0, t_1\}$ and a linear map $S^{m-1}U \oplus S^{n-1}U \xrightarrow{M_{A,B}} S^{m+n-1}U$ which sends a pair of polynomials $(h_1(t), h_2(t))$ to $A(t)h_1(t) + B(t)h_2(t)$ as in 8-2. The Silvester matrix is transpose

[1]the leading terms of their expansions as polynomials in d are $md^n/n!$ and $d^n/n!$ respectively

to the matrix of $M_{A,B}$ in the standard monomial bases. If a point (α,β) lies on the quadric $\alpha'_i\beta''_j - \alpha''_i\beta'_j = 0$, then $(\alpha''_i t_0 - \alpha'_i t_1) = (\beta''_i t_0 - \beta'_i t_1)$ up to a scalar factor. This linear form divides $A(t)$, $B(t)$ and any polynomial $A(t)h_1(t) + B(t)h_2(t)$. Hence, $\operatorname{im} M_{A,B} \neq S^{m+n-1}U$. So, the Silvester determinant vanishes along each quadric $\alpha'_i\beta''_j - \alpha''_i\beta'_j = 0$ and, by Hilbert's Nullstellensatz, it is divisible by $R_{A,B}$. The quotient has to be a constant, because the both polynomials are bihomogeneous of the same bidegree (mn, mn) in (α,β). Moreover, since the Silvester determinant vanishes as soon $A(t)$ and $B(t)$ are not coprime, R_{AB} spans the ideal of all resultant relations — so, it is principal in the case of two binary forms.

8.8.3. Example: elimination technique. Let C_1, C_2 be two plane curves of degrees m and n given by equations $F(x) = 0$ and $G(x) = 0$ in $x = (x_0 : x_1 : x_2)$. Consider F, G as (non homogeneous) polynomials in x_0 with the coefficients in $\Bbbk(x_1, x_2)$ and take their resultant[1] $R_{F,G}(x_1, x_2) \in \Bbbk[x_1, x_2]$. If it is identically zero, then F and G have a common divisor in $\Bbbk(x_1, x_2)[x_0]$.

Exercise 8.6. Deduce from the Gauss lemma that it can be taken with the coefficients in $\Bbbk[x_1, x_2]$.

So, if $R \equiv 0$, then C_1 and C_2 have a common component. If $R \not\equiv 0$, then (x_1, x_2)-coordinates of any intersection point $p \in C_1 \cap C_2$ have to satisfy the resultant equation $R_{F,G}(x_1, x_2) = 0$ which is homogeneous of degree mn. So, the curves have either a common component or at most mn intersection points, which may be found by solving a homogeneous polynomial equation in x_1, x_2 only. These procedure is called *an elimination of a variable*.

[1]resultant of non-homogeneous polynomials $F(x_0)$ and $G(x_0)$ is defined as the resultant of $t_0^n F(t_1/t_0)$ and $t_0^m G(t_1/t_0)$

§9. Projective hypersurfaces.

In this section we assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk \neq 2$.

9.1. Space of hypersurfaces. Projective space $\mathbb{P}(S^dV^*)$ consists of all non zero d-th degree homogeneous polynomials considered up to a scalar factor. It is called *a space of degree d hypersurfaces* in $\mathbb{P}_n = \mathbb{P}(V)$. Geometrically, each polynomial whose prime factorization is $f(x) = \prod_{i=1}^{s} p_i(x)^{m_i}$ defines a zero set

$$Z_f \overset{\text{def}}{=} \{x \in \mathbb{P}_n \mid f(x) = 0\} = \bigcup_{i=1}^{s} m_i \cdot Z_{p_i} \,,$$

which is an union of the *irreducible components* $Z_{p_i} = \{x \in \mathbb{P}(V) \mid p_i(x) = 0\}$ counted with integer multiplicities m_i. We will also write $Z_f = m_1 Z_{p_1} + m_2 Z_{p_2} + \dots + m_s Z_{p_s}$. By ex. 8.5, each irreducible component Z_{p_i} does not admit any further decomposition into a sum of proper subsurfaces.

Exercise 9.1. Find $\dim \mathbb{P}(S^dV^*)$.

Traditionally, 1-, 2-, and 3-dimensional projective subspaces in the space of hypersurfaces are called, respectively, *pencils*, *nets* and *webs* of hypersurfaces.

9.1.1. Example: pencil of plane curves $\ell = (C_1C_2) \subset \mathbb{P}(S^dV^*)$ on $\mathbb{P}_2 = \mathbb{P}(V)$ is defined by any two distinct elements $C_1, C_2 \in \ell$. A curve $\lambda C_1 + \mu C_2 \in \ell$ (whose homogeneous coordinates w.r.t. the basis $\{C_1, C_2\}$, of ℓ, are $(\lambda : \mu)$) is given in $\mathbb{P}_2$ by the equation $\lambda f_1(x) + \mu f_2(x) = 0$, where $f_1(x) = 0$ and $f_2(x) = 0$ are the equations of the basic curves C_1, C_2. In particular, each curve from the pencil (C_1C_2) contains all intersection points $C_1 \cap C_2$. Another remarkable property: any pencil of plane curves contains a curve passing through any prescribed point $p \in \mathbb{P}_2$. Indeed, curves passing through a given point form a codimension 1 hyperplane in the space of curves and this hyperplane intersects each line of curves.

As an application of pencils, let us give another fruitful proof of the Pascal theorem from n° 3.3.1. Given a hexagon $p_1p_2p_3p_4p_5p_6$ inscribed in a non singular conic C, write $x = p_3p_4 \cap p_6p_1$, $y = p_2p_3 \cap p_5p_6$, $z = p_1p_2 \cap p_4p_5$ for the intersection points of its opposite sides. Fix some 7-th point $p_7 \in C$, which differs from $p_1, p_2, \dots, p_6$, and consider a pencil of cubic curves (Q_1Q_2) spanned by 2 completely splitted cubics formed by 'opposite triples' of sides $Q_1 = (p_1p_2) \cup (p_3p_4) \cup (p_5p_6)$ and $Q_2 = (p_2p_3) \cup (p_4p_5) \cup (p_6p_1)$. All cubics of this pencil pass through 9 intersection points $Q_1 \cap Q_2 = \{p_1, p_2, \dots, p_6, x, y, z\}$ and at least one of them, say Q, pass through p_7 as well. Since the conic C has more than 6 common points with the cubic Q, it should be a component of this cubic, i.e. $Q = C +$ some line, where the line has to pass through $x, y, z \notin C$.

9.2. Interaction with lines. Let $S \subset \mathbb{P}_n$ be a hypersurface given by a homogeneous equation $F(x) = 0$ of degree d and $\ell = (pq) \in \mathbb{P}_n$ be a line spanned by $p, q \in V$. Write $(\lambda : \mu)$ for internal homogeneous coordinates of a point $\lambda p + \mu q \in \ell$. In these coordinates, $\ell \cap S$ is given by the equation $f(\lambda, \mu) = 0$ obtained from $F(x) = 0$ by the substitution $x = \lambda p + \mu q$. By the Newton–Taylor formula,

$$f(\lambda, \mu) = F(\lambda p + \mu q) = \sum_{i=0}^{d} \lambda^i \mu^{n-i} \binom{d}{i} \widetilde{F}(p^i, q^{n-i}) \,, \quad \text{where} \tag{9-1}$$

$$\widetilde{F}(p^i, q^{n-i}) \overset{\text{def}}{=} \operatorname{pl}(F)(\underbrace{p, p, \dots, p}_{i}, \underbrace{q, q, \dots, q}_{d-i}) = \frac{(d-i)!}{d!} \frac{\partial^i F}{\partial p^i}(q) = \frac{i!}{d!} \frac{\partial^{d-i} F}{\partial q^{d-i}}(p) \,. \tag{9-2}$$

Note that the bottom term $F(p^i, q^{n-i})$ is bihomogeneous of degree $(i, n-i)$ in (p, q).

If $f(\lambda, \mu) \equiv 0$ or, equivalently, $\widetilde{F}(p^i, q^{n-\mu}) = 0$ for all i, then $\ell \subset S$.

If $f(\lambda, \mu) \not\equiv 0$, then $f(\lambda, \mu) = \prod_i (\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is a product of linear forms[1]. Each linear form corresponds to an intersection point $\alpha = (\alpha' : \alpha'') = \alpha' p + \alpha'' q \in \ell \cap S$. The maximal power s_i such that $f(\lambda, \mu)$ is divisible by $(\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is called *a local intersection index* between S and ℓ at α. It is denoted by $(S, \ell)_\alpha$. So, $\deg S = \sum_{\alpha \in S \cap \ell} (S, \ell)_\alpha$ as soon as $\ell \not\subset S$, i.e. a line either lies on S or intersect S in $\deg S$ points counted with multiplicities.

9.3. Tangent lines and tangent space. A line ℓ is called *a tangent line* to S, if there is a point $p \in S \cap \ell$ with $(S, \ell)_p \geqslant 2$. We say that ℓ does *touch* S at p or that p is a *tangency point*.

[1] namely, $f(\lambda, \mu) = \mu^d f(t, 1)$, where $t = \lambda/\mu$ and $f(t, 1) \in \Bbbk[t]$; now, $f(t, 1) = \prod (t - \alpha_i)^{m_i}$

9.3.1. CLAIM. *For any $p \in S$ and any $q \in \mathbb{P}_n$ the line (pq) touch S at p iff $\widetilde{F}(p^{n-1}, q) = 0$.*

Proof. If $p \in S$, that is $\widetilde{F}(p^n) = F(p) = 0$, the affine version of (9-1) near p takes the form:

$$F(p + tq) = t \binom{d}{1} \widetilde{F}(p^{n-1}, q) + t^2 \binom{d}{2} \widetilde{F}(p^{n-2}, q^2) + \cdots$$

and $(S, (pq))_p$ is the maximal power of t factored out of $F(p + tq)$. It is $\geqslant 2$ iff $\widetilde{F}(p^{n-1}, q) = 0$. □

9.3.2. COROLLARY. *The union of all tangent lines through $p \in S$ is a projective space*

$$T_p S \overset{\text{def}}{=} \left\{ y \in \mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} y_i \frac{\partial F}{\partial x_i}(p) = 0 \right\} .$$

It is either a hyperplane or the whole of $\mathbb{P}_n$. The last happens iff $\dfrac{\partial F}{\partial x_i}(p) = 0 \; \forall i$. □

The space $T_p S$ is called *a tangent space* to S at p. If $T_p S = \mathbb{P}_n$, then S is called *singular* at p and p is called *a singular point* of S. Otherwise p is called *a smooth point* of S. S is called *smooth*, if all its points are smooth.

9.3.3. COROLLARY. *Let q be either a smooth point on S or any point outside S. Then the apparent contour*[1] *of S visible from q is slashed by the hypersurface of degree $(d-1)$*

$$S_q^{(d-1)} \overset{\text{def}}{=} \left\{ y \in \mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} q_i \frac{\partial F}{\partial x_i}(y) = 0 \right\} .$$

In particular, $\sum\limits_{i=0}^{n} q_i \dfrac{\partial F}{\partial x_i}(y) \not\equiv 0$ as a polynomial in y.

Proof. Indeed, (qy) touch S at y, if $0 = \widetilde{F}(y^{n-1}, q) = \mathrm{pl}_q F(y) = \frac{1}{d} \sum_{i=0}^{n} q_i \dfrac{\partial F}{\partial x_i}(y)$. If this polynomial vanishes identically in y, then taking $y = q$ we get $F(q) = 0$, i. e. $q \in S$. At the same time

$$F(q, q, \ldots, q, y) = \mathrm{pl}_q^{n-1} F(y) = \mathrm{pl}_q^{n-2} \mathrm{pl}_q F(y) \equiv 0 ,$$

because of $\widetilde{F}(y^{n-1}, q) \equiv 0$. So, q is singular point of S. □

9.4. Point multiplicities. A number $\mathsf{mult}_S(p) \overset{\text{def}}{=} \min\limits_{\ell \ni p} (\ell, S)_p$ is called *a multiplicity* of p on S. A point $p \in S$ is singular iff any line through p intersects S with index $\geqslant 2$ at p. So, $p \in S$ is smooth iff p has the multiplicity 1. A point p has multiplicity $\geqslant m$ iff all possible $(m-1)$-typle partial derivatives of F vanish at p.

9.5. Polar hypersurfaces. A hypersurface $S_q^{(r)} \overset{\text{def}}{=} \left\{ y \in \mathbb{P}_n \;\middle|\; \widetilde{F}(q^{n-r}, y^r) = 0 \right\}$ is called *a r-th degree polar* of S with respect to p. If $F(q^{n-r}, y^r)$ vanishes identically in y, we say that the polar is *trivial*, i. e. coincides with the whole of $\mathbb{P}_n$. Intuitively, for a smooth point $q \in S$, the polar $S_q^{(r)}$ is a degree r surface which gives the most closed approximation for S near q in a sense that the both have at q the same tangent hyperplanes (i. e. their linear polars at q coincide), the same 'tangent quadrics' (i. e. their quadratic polars at q coincide), and so on up to coincidence of $(r-1)$-th degree polars. If $q \in S$ is singular of multiplicity $m \geqslant 2$, then all the polars of degree $\leqslant (m-1)$ w. r. t. p are trivial and the m-th degree polar is non trivial but singular at p.

9.5.1. Example: space of singular conics. Let V be 3D vector space, $\mathbb{P}_5 = \mathbb{P}(S^2 V^*)$ be the space of conics on $\mathbb{P}_2 = \mathbb{P}(V)$, and $S \subset \mathbb{P}_5$ be a locus of the singular conics. Let us fix some coordinates and present quadratic forms $q(x) \in S^2 V^*$ as $q(x) = x \cdot A \cdot {}^t x$ with symmetric 3×3-matrices A. Since q is singular iff $\det A = 0$, we see that S is an irreducible cubic hypersurface in $\mathbb{P}_5$. We would like to find its singular points and describe non singular tangent hyperplanes. By Sylvester's relations, $\det A = \sum\limits_{\nu} (-1)^{i+\nu} a_{i\nu} A_{i\nu}$, where $A_{i\nu}$ is 2×2-minor situated outside i-th row and j-th column. So, $\dfrac{\partial \det A}{\partial a_{ij}} = (-1)^{i+j} A_{ij}$ and a point $q \in S$ is singular iff $\operatorname{rk} A = 1$.

Exercise 9.2. Show that any $m \times n$ matrix a_{ij} of rank 1 has $a_{ij} = \lambda_i \mu_j$, i. e. can be written as the product of appropriate column ${}^t(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and row $(\mu_1, \mu_2, \ldots, \mu_n)$.

[1] i. e. the set of all tangency points $p \neq q$ where S touched by the tangent lines drawn from q

HINT. A linear operator $k^n \xrightarrow{x \mapsto Ax} k^m$ has $\operatorname{rk} A = 1$ iff $\dim \operatorname{im} A = 1$; if $w = (\lambda_1, \lambda_2, \dots, \lambda_m)$ generates $\operatorname{im} A$, then $A(v) = \alpha(v)\, w$, where $k^n \xrightarrow{\alpha} k$ is linear form, say $\alpha = (\mu_1, \mu_2, \dots, \mu_n) \dots$

In our case $A = (a_{ij})$ is symmetric and we should have $\mu_i = \lambda_i$, i.e. $a_{ij} = \lambda_i \lambda_j$ for some $\lambda_0, \lambda_1, \lambda_2 \in k$. So, $A \in S$ is singular iff $q(x) = \left(\sum \lambda_i x_i\right)^2$ is a double line. Thus, the set of singular points of S coincides with 2-dimensional Veronese's surface $\mathscr{V} \subset S$, which parameterizes double lines in $\mathbb{P}_2$.

Now, let $q(x) = x \cdot A \cdot {}^t x$ be a smooth point of S, i.e. a pair of distinct lines $\ell_1 \cup \ell_2 \subset \mathbb{P}_2$. Then the correlation map $V \xrightarrow{x \mapsto x \cdot A} V^*$ has 1-dimensional kernel spanned by $v = \ell_1 \cap \ell_2$, that is $\operatorname{rk}(A) = 2$ and the adjoint matrix $\widehat{A} = \big((-1)^{i+j} A_{ij}\big)$ is non zero. By the Sylvester relations: $A \cdot \widehat{A} = \widehat{A} \cdot A = \det(A) \cdot \mathrm{Id} = 0$, each row and each column of $\widehat{A}$ lies in the kernel of A, i.e. is proportional to v. Thus, $\operatorname{rk} \widehat{A} = 1$ and $(-1)^{i+j} A_{ij} = \mu_i \mu_j$, where $(\mu_0 : \mu_1 : \mu_2)$ are homogeneous coordinates of v. So, $B = (b_{ij}) \in T_q S_n \iff \sum_{ij} b_{ij} \cdot (-1)^{i+j} A_{ij} = 0 \iff \sum_{ij} b_{ij}\, \mu_i \mu_j = 0 \iff v \cdot B \cdot {}^t v = 0$. In other words, the tangent space $T_q S$ at $q = \ell_1 \cup \ell_2 \subset \mathbb{P}_2$ consists of all conics passing through the point $\ell_1 \cap \ell_2 \in \mathbb{P}_2$.

Exercise 9.3. Extend this result to general case $\dim V = n + 1$, i.e. show that a point q on the hypersurface $S \subset \mathbb{P}(S^2 V^*)$, of singular quadrics on $\mathbb{P}_n = \mathbb{P}(V)$, is non-singular iff the corresponding quadric $Q_q \subset \mathbb{P}(V)$ has just one singular point $v(q) \in \mathbb{P}(V)$ and prove that $T_q S$ consists of all quadrics passing through $v(q)$.

§10. Working example: plane curves.

In this section we assume that $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk \neq 2$.

10.1. Geometrical tangents at singularity. Let $C \subset \mathbb{P}_2$ be a curve given by an equation $F(x) = 0$ of degree d and $p \in C$ be a (singular) point of multiplicity $m \geqslant 2$. Then all the polars $C_p^{(\nu)}$ (which would be given by equations[1] $\widetilde{F}(p^{d-\nu}, x^{\nu}) = 0$) are trivial for $0 \leqslant \nu \leqslant (m-1)$ and m-th degree polar $C_p^{(m)}$ (given by $\widetilde{F}(p^{d-m}, x^m) = 0$) is non trivial but singular: its Taylor expansion near p

$$\widetilde{F}(p^{d-m}, (p+tq)^m) = \sum_{\mu=1}^{m} t^{\mu} \binom{m}{\mu} \widetilde{F}(p^{d-m+\mu}, q^{\mu}) = t^m \, \widetilde{F}(p^{d-m}, q^m)$$

contains just one term and a line (p,q) either is a component of $C_p^{(m)}$ (when $\widetilde{F}(p^{d-m}, q^m) = 0$) or intersects $C_p^{(m)}$ only at p with multiplicity m (when $F(p^{d-m}, q^m) \neq 0$). So, $C_p^{(m)}$ splits into union of m lines (pq_i), where q_i are the roots of $\widetilde{F}(p^{d-m}, q^m) = 0$ considered as degree m equation on q, where q runs through any fixed line $\ell \not\ni p$. (Of course, some of (pq_i) may coincide when the roots became multiple.) The lines (pq_i) are called *geometrical* tangent lines to C at p.

Geometrically, generic line (pq), through p, intersects C at p with multiplicity m, because the Taylor expansion $F(p+tq) = \binom{d}{m} \cdot t^m \cdot \widetilde{F}(p^{d-m}, q^m) + \cdots$ starts with non-zero m-th degree term. The geometric tangents (pq_i) are the lines whose intersection multiplicity with C at p jumps w. r. t. the generic value. Algebraically, this means that $\widetilde{F}(p^{d-m}, x^m) = \xi_1(x)\,\xi_2(x) \cdots \xi_m(x)$ is the product of m linear forms $\xi_1, \xi_2, \dots, \xi_s$ whose zeros are the geometric tangents (pq_i) (again, some of them may coincide).

10.1.1. Example: the simplest singularities. Given a curve $C \subset \mathbb{P}_2$, an m-typle point $p \in C$ is called *an m-typle node* (or *an m-typle selfintersection*) if there are m distinct geometrical tangents $\ell_1, \ell_2, \dots, \ell_m$ to C through p. Geometrically, this means that C has m mutually transversal branches through p. The difference $(\ell_i, C)_p - m - 1$ is called *an order of the contact* between ℓ_i and the corresponding branch of C. A node is called *ordinary* if all the geometrical tangents have the second order contacts with its branches, that is $(\ell_i, C)_p = m + 1 \ \forall\, i$.

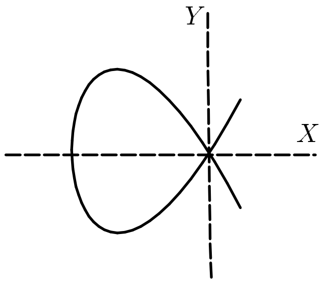


Fig. 10◇1. The node $y^2 = x^2(x+1)$.

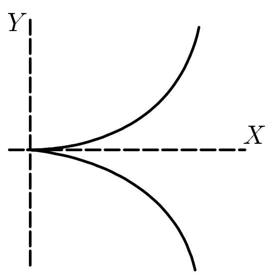


Fig. 10◇2. The cusp $y^2 = x^3$.

A double point $p \in C$ is called *a cusp* (or *a selfcontact*) if the quadratic polar of p is a double line ℓ. Geometrically, this means that C has two branches which do touch each other at p. The unique geometrical tangent ℓ at p is called *a cuspidal tangent*. A local intersection number $(\ell, C)_p$ measures *an order of the selfcontact* for C at p; clearly, $(\ell, C)_p \geqslant 3$. A cusp is called *ordinary*, if $(\ell, C)_p = 3$ is minimal possible.

We say that C has *only the simplest singularities*, if the singular points of C are exhausted by ordinary double nodes and ordinary cusps. Two cubic curves with the simplest singularities are shown on the figs. fig 10◇1–fig 10◇2. For higher degree curves, the neighborhood of the simple singularity looks similarly[2].

Exercise 10.1. Show that irreducible cubic curve has at most one (automatically simple) singularity.

HINT. A line ℓ has to be a component of a cubic C as soon as $(\ell, C) \geqslant 4$.

10.1.2. Example: how much is to put a singularity on a curve? Given a point $p \in \mathbb{P}_2$, then the polar map

$$S^d V^* \xrightarrow{F \mapsto \partial^{d-m} F / \partial p^{d-m}} S^m V^*$$

[1] recall that $\widetilde{F}(p^{d-\nu}, x^{\nu}) = \mathrm{pl}(F)(\underbrace{p, p, \dots, p}_{d-\nu}, \underbrace{x, x, \dots, x}_{\nu}) = \dfrac{\nu!}{d!} \dfrac{\partial^{(d-\nu)} F}{\partial p^{(d-\nu)}}(x) = \dfrac{(d-\nu)!}{d!} \dfrac{\partial^{\nu} F}{\partial x^{\nu}}(p)$

[2] one can show that any smooth curve in $\mathbb{P}_n$ admits a plane projection that has only the simplest singularities

is a linear epimorphism. Hence the curves whose m-degree polar coincides with a given collection of lines through p form a projective subspace of codimension $\dim(S^m V^*) - 1 = m(m+3)/2$. For example, 5 parameters in a curve equation are fixed by assuming that this curve has at a given point a cusp with a given cuspidal tangency. However, these restrictions, if come from several distinct points, are not independent, in general.

10.2. Affine neighborhood of a singularity. Practical computation of geometric tangents usually becomes simpler in affine chart with the origin in the singular point in question. Let C have an affine equation $f(x,y) = 0$ in such a chart U. Write it as $\sum\limits_{\nu>0} f_\nu(x,y) = 0$, where each $f_\nu(x,y)$ is homogeneous of degree ν, and consider a line $\ell_{\alpha:\beta}$ given parametrically as[1] $x = \alpha t\,,\ y = \beta t$. Then $C \cap \ell$ is given by the following equation on t:

$$f_m(\alpha,\beta)\,t^m + f_{m+1}(\alpha,\beta)\,t^{m+1} + \cdots + f_d(\alpha,\beta)\,t^d = 0$$

where m is the degree of lowest non trivial homogeneous component of f and each $f_\nu(\alpha,\beta)$ is actually nothing but the ν-th degree polar of p evaluated at $q = (\alpha : \beta)$. Thus, the multiplicity of p coincides with the degree of lowest non trivial homogeneous component of f and the directions $(\alpha : \beta)$ of the geometrical tangents through p are the roots of this component, i.e. satisfy the equation $\varphi_m(\alpha,\beta) = 0$.

10.2.1. Example: analyzing singularities. Taking $x = \alpha t$, $y = \beta t$ in the nodal cubic equation $y^2 - x^2 - x^3 = 0$, we get the lowest term $(\beta+\alpha)(\beta-\alpha)\,t^2$, which vanishes for $(\alpha:\beta) = (1 : \pm 1)$; so, there are two distinct geometrical tangents $x = \pm y$. Local intersection number $(\ell, C)_p = 3$ for each tangent line ℓ, i.e. each tangent has the second order contact with its branch. The cuspidal cubic on the fig 10◇2 has the lowest term $\beta^2\,t^2$. So, the second polar is a the double line $x = 0$ with local intersection 3 with C at the origin.

As an advanced example, consider a quartic given by the polynomial $F(t) = t_0^4 - t_0^3 t_1 + t_0^2 t_2^2 - t_1^2 t_2^2$. Its singularities are $t \in \mathbb{P}_2$ where all partial derivatives

$$\begin{aligned} \partial F/\partial t_0 &= 4\,t_0^3 - 3\,t_0^2 t_2 - 2\,t_0 t_2^2 \\ \partial F/\partial t_1 &= -2\,t_1 t_2^2 \\ \partial F/\partial t_2 &= -t_0^3 + 2\,t_0^2 t_2 - 2\,t_1^2 t_2 \end{aligned}$$

vanish simultaneously. It happens at two points $a = (0:0:1)$ and $b = (0:1:0)$. Take an affine chart with $x = t_0/t_2\,,\ y = t_1/t_2$ near a. Then $F = 0$ turns into $x^2 - y^2 - x^3 + x^4 = 0$ with two simple geometrical tangents $x = \pm y$ at the origin. Since a local intersection number equals 3 for each tangency, a is an ordinary node. Taking a chart with $x = t_0/t_1\,,\ y = t_2/t_1$ near b, we get the equation $y^2 - x^4 + x^3 y - x^2 y^2$ whose geometrical tangent is a double line $\ell = \{x = 0\}$ with $(\ell, C)_p = 4$. So, b is the non ordinary cusp, where C has a selfcontact of order 4.

10.3. Blow up. Geometrically, the substitution $x = \alpha t$, $y = \beta t$ lifts C from $\mathbb{P}_2$ to a surface $\Gamma \subset \mathbb{P}_1 \times \mathbb{P}_2$ called *a blow up* of $p \in \mathbb{P}_2$. It is described as follows. Identify a pencil of lines through p with any fixed line $\mathbb{P}_1 = (ab) \not\ni p$ and consider the incidence graph $\Gamma \overset{\text{def}}{=} \{(\ell, q) \in \mathbb{P}_1 \times \mathbb{P}_2 \mid \ell \ni q\}$. It is an algebraic surface in $\mathbb{P}_1 \times \mathbb{P}_2$: if we put $p = (1:0:0)$, $a = (0:1:0)$, $b = (0:0:1)$, take $q = \alpha a + \beta b$, and consider $((\alpha:\beta), (x_0 : x_1 : x_2))$ as coordinates on $\mathbb{P}_1 \times \mathbb{P}_2$, then $(x_0 : x_1 : x_2) \in (pq)$ is equivalent to the quadratic relation $\alpha x_2 = \beta x_1$.

Projection $\Gamma \xrightarrow{\sigma} \mathbb{P}_2$ is bijective outside p, but $\sigma^{-1}(p) \simeq \mathbb{P}_1$ is the pencil of lines through p on $\mathbb{P}_2$ (see fig 10◇3). A map $(\,(\alpha:\beta)\,,\, t) \longmapsto (\,(\alpha:\beta)\,,\,(1:\alpha t:\beta t)\,) \in \mathbb{P}_1 \times \mathbb{P}_2$ gives a rational parameterization for some affine neighborhood of this exceptional fiber in Γ. Full preimage $\sigma^{-1}(C)$, of a curve $C \subset \mathbb{P}_2$ passing through p, consists of 2 components: $\sigma^{-1}(p) \simeq \mathbb{P}_1$ and and some curve whose equation (in terms of parameters $(\alpha:\beta;t)$ on Γ) is a result of the substitution $x = \alpha t$, $y = \beta t$ in the affine equation for C.

[1] as in the beginning of this lecture, line $\ell_{\alpha:\beta}$ has a form $(p + tq)$, where $q = (\alpha:\beta) \in U_\infty \simeq \mathbb{P}_1$ is runing through the infinite projective line of the chart U

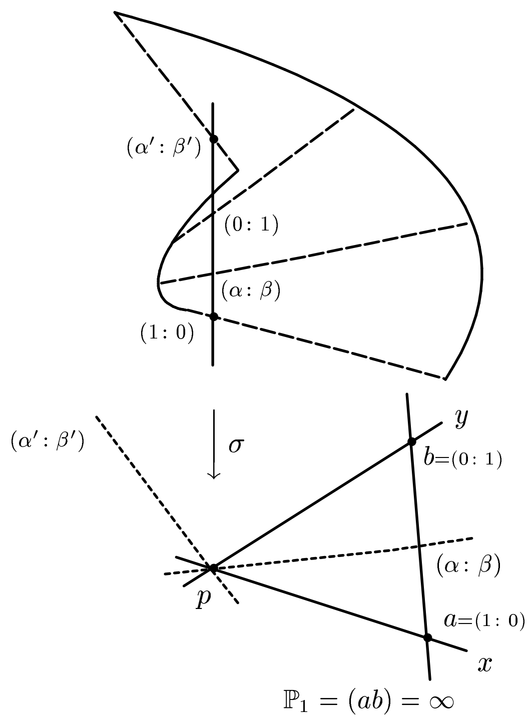


Fig. 10◇3. Blow up.

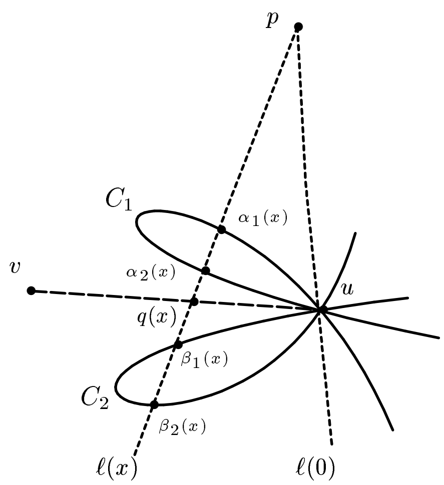


Fig. 10◇4. The Zeuthen rule.

10.4. Local intersection multiplicity. Consider two curves $C_1, C_2 \subset \mathbb{P}_2$ given by homogeneous equations $F = 0$, $G = 0$ of degrees n, m without common divisors. Let $u \in C_1 \cap C_2$. We fix two points p, v such that the line (pu) satisfies the conditions:

$$(pu) \cap C_1 \cap C_2 = \{u\} \qquad \& \qquad v \notin (pu) \tag{10-1}$$

and will use the triple $\{p,\, u,\, v\}$ as a basis for $\mathbb{P}_2$. Let a point $q(x) = u + x\,v$ tend to u (as $x \to 0$) along the line (pu). We restrict the both curves onto a varying line $\ell(x) = (p, q(x))$ and write $\alpha_0(x), \dots, \alpha_n(x)$ for the points $C_1 \cap \ell(x)$ and $\beta_0(x), \dots, \beta_m(x)$ for the points $C_2 \cap \ell(x)$ (see fig 10◇4). These points are the roots of two homogeneous polynomials $f_x(t_0, t_1) = F(t_0 p + t_1 q(x))$, $g_x(t_0, t_1) = G(t_0 p + t_1 q(x))$ in $t = (t_0 : t_1)$ with coefficients depending on the parameter x. When x varies, the pints $\alpha_\nu(x)$, $\beta_\mu(x)$ draw the branches of C_1 and C_2. Let $\beta_1(x), \dots, \beta_r(x)\,;\ \alpha_1(x), \dots, \alpha_s(x)$ be all the branches that come to u as $x \to 0$.

Intuitively, over «a continuous field» like $\Bbbk = \mathbb{R}, \mathbb{C}$, a local intersection multiplicity $(C_1, C_2)_u$, of the curves at the point u, equals to the sum of orders of rs infinitesimals $\alpha_i(x) - \beta_j(x)$ w. r. t. x as $x \to 0$. This naive geometric definition is known as *the Zeuthen rule*.

Algebraically, the sum of orders of infinitesimals $\alpha_i(x) - \beta_j(x)$ coincides with the multiplicity of the zero root of the resultant R_{f_x, g_x} considered as a polynomial in x. Thus, over an arbitrary field $\Bbbk$ we can define $(C_1, C_2)_u$ as the multiplicity of the factor x in the prime factorization of R_{f_x, g_x} in $k[x]$. This definition does not depend on a choice of p, v satisfying (10-1), because of the following

10.4.1. LEMMA. *Consider the resultant $R_{f,g}(p, q) \in k[p, q]$, of two binary forms $f(t_0, t_1) = F(t_0 p + t_1 q)$, $g(t_0, t_1) = G(t_0 p + t_1 q)$, as a polynomial in $p = (p_0, p_1, p_2)$, $q = (q_0, q_1, q_2)$. Then its irreducible factorization in $k[p, q]$ has a form*[1]*:*

$$R_{f,g}(p, q) = const \cdot \prod_{w \in C_1 \cap C_2} \det \begin{pmatrix} w_0 & w_1 & w_2 \\ p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}^{m_w} \tag{10-2}$$

where the multiplicities m_w are computed by the Zeuthen rule with any choice of p, v satisfying (10-1)*.*

[1] geometrically, $R_{f,g}(p, q) = 0$ defines in the space of lines (pq) a figure whose irreducible components are pencils of lines centered at the intersection points and the multiplicities of these components are predicted by the Zeuthen rule

Proof. Denote the determinants in the right side of (10-2) by $D_w(p,q)$. Geometrically, $D_w(p,q) = 0$ is an irreducible quadric, which consists of all pairs $p, q \in \mathbb{P}_2$ whose joining line (pq) belongs to the pencil of lines through w. If $w \in C_1 \cap C_2$, then $R(p,q)$ vanishes along the quadric $D_w(p,q) = 0$. Hence, by Hilbert's Nullstellensatz, any such $D_w(p,q)$ divides the resultant. Vice versa, if $R(p,q) = 0$, then the restrictions of F, G onto the line (p,q) have a common root, i. e. (p,q) pass through some $w \in C_1 \cap C_2$ and $\prod_{w\in C_1\cap C_2} D_w(p,q)$ vanishes at this (p,q). So, this product vanishes everywhere along $V(R(p,q))$ and, again by Hilbert, $R(p,q)$ should divide some power of $\prod_{w\in C_1\cap C_2} D_w(p,q)$. To check the Zeuthen rule, fix p and $q = q(x)$ as on fig 10◇4. The condition (10-1) implies that only $D_w(p,q(x))$ with $w = u$ vanishes at $x = 0$ in the right hand side of (10-2). This vanishing determinant $D_u(p,q(x)) = \det(u,p,u + x\,v) = x\,\det(u,p,v)$ is proportional to x. □

10.5. Intersection theory of plane curves. It follows immediately from the Zeuthen rule, that the local intersection multiplicities are distributive w. r. t. the curve branches, that is if C_1 has b_1 branches passing through u and C_2 has b_2 ones, then $(C_1,C_2)_u$ is the sum of b_1b_2 mutual intersection indices between the branches[1].

Since each $D_w(p,q)$ in (10-2) is bilinear in (p,q) and $R_{f,g}(p,q)$ has bidegree (mn,mn), we get *the Bézout theorem*:

10.5.1. THEOREM. $\sum\limits_{w\in C_1\cap C_2} (C_1,C_2)_w = \deg C_1 \cdot \deg C_2$ *for any two plane projective curves without common components.* □

10.5.2. Example: proper tangents and class. A tangent lane is called *proper*, if its tangency point is smooth. A number of proper tangents to C passing through a generic point $q \in \mathbb{P}_2$ is called *a class* of C and denoted by $c = c(C)$. If $\deg C = d$, then by n° 9.3.3 the tangents coming form a point $q \in \mathbb{P}_2 \setminus \mathrm{Sing}\,(C)$ touch C at the points of $C \cap C_q^{(d-1)}$, where $C_q^{(d-1)}$ is $(d-1)$-th degree polar of q. If C is irreducible, then $C \cap C_q^{(d-1)}$ consists of $d(d-1)$ points[2] counted with multiplicities. Besides the proper tangency points, $C \cap C_q^{(d-1)}$ contains also all singular points of C, because each line trough a singularity is (non proper) tangent. So, class of irreducible curve satisfies inequality $c \leqslant d(d-1)$, which turns to equality iff C is smooth.

10.5.3. Example: inflections. A smooth point $p \in C$ is called *an inflection*, if $(C, T_pC)_p \geqslant 3$. An inflection is called *ordinary* (or *simple*), if this number equals 3. If $p \in C$ is an inflection, then the quadratic polar $C_p^{(2)}$ of p has the zero restriction onto $\ell = T_pC$, i. e. ℓ is a component of $C_p^{(2)}$. Note that p is a smooth point of the conic $C_p^{(2)}$, because $C_p^{(2)}$ and C have the same linear polar w. r. t. p and p is smooth on C. So, $p \in C$ is an inflection iff $C_p^{(2)} = \ell \cup \ell'$ with $\ell \cap \ell' \neq p$. The points $q \in \mathbb{P}_2$ with degenerate quadratic polar $C_q^{(2)}$ form a curve H_C, which is called *the Hessian* of C. It is defined by the equation $\det C_q^{(2)} = 0$, which has degree $3(d-2)$ in q, where $d = \deg C$. Hence, an irreducible curve of degree $d \geqslant 3$ has at most $3d(d-2)$ inflections, which are contained in $C \cap H_C$. Again, this intersection contains also all singular points[3] of C.

10.5.4. Example: affine localization. Let us restrict the picture fig 10◇4 onto affine chart where (pv) is the infinity, (uv) is the x-axis, and (up) is the y-axis. Then the line pencil through p turns to the family of vertical lines $x = \text{const}$. Consider affine equations $f(x,y) = 0$, $g(x,y) = 0$ for C_1, C_2 as (nonhomogeneous) polynomials in y with the coefficients in $k[x]$. Their resultant $R_{f,g}(x)$ is a polynomial in x and vanishes at $x = 0$. The multiplicity of this zero root coincides with $(C_1,C_2)_{(0,0)}$. If there are known some explicit analytic expressions of all the branches $y = \alpha_i(x)$ and $y = \beta_j(x)$ through x (even not algebraic, say several starting terms of the (formal fractional) power series expansions are OK), then $(C_1,C_2)_{(0,0)}$ usually can be also computed explicitly by looking at either the orders of $\alpha_i(x) - \beta_j(x)$ or the order of the resultant.

10.6. Dual curves. Let $C \subset \mathbb{P}_2$ be an irreducible curve given by an equation $F(x) = 0$ of degree d. For any smooth $p \in C$ its tangent $\tau_p = T_pC$ defines a point $\tau_p^* = \mathrm{Ann}\,\tau_p \in \mathbb{P}^\times$ on the dual plane $\mathbb{P}_2^\times$. When p varies along C, τ_p^* also is running through some curve $C^\times \subset \mathbb{P}_2^\times$ called *a dual curve* for C. The degree of dual curve, i. e. the number of its intersection points with a generic line $\vartheta = q^\times \subset \mathbb{P}_2^x$, is nothing but the

[1] in particular, $(n_1C_1 + n_2C_2, D) = n_1(C_1,D) + n_2(C_2,D)$, where $m_1C_1 + m_2C_2$ is a curve given by equation $F_1^{m_1}F_2^{m_2} = 0$ and $F_1 = 0$, $F_2 = 0$ are the equations for C_1 and C_2

[2] Note that this is not true in positive characteristic: for example, if $\mathrm{char}\,(k) = 2$, then *all* tangents to the smooth conic $x_0^2 = x_1x_2$ pass through one common point.

[3] In fact, even for singular curves one can write precise equations between degree, class, number of inflections and some data describing singularities; we'll do this below for curves with simplest singularities.

number of proper tangents to C living in a generic pencil of lines (centered at a generic point $q \in \mathbb{P}_2$). Thus $\deg(C^\times) = c(C)$.

10.6.1. CLAIM. *$C^{\times\times} = C$; in particular,* $\deg(C) = c(C^\times)$.

Proof. A tangent line $\vartheta = T_{\tau_1^*}C^\times \subset \mathbb{P}_2^\times$, at a smooth point $\tau_1^* \in C^\times$, is a limit of secant lines $\sigma = (\tau_1^*, \tau_2^*)$ as $\tau_2^* \to \tau_1^*$ (see fig 10◇5 – fig 10◇6). The secant σ represents a pencil of lines through $\sigma^* = \tau_1 \cap \tau_2 \in \mathbb{P}_2$. Clearly, $\sigma^* \to p_1$ as $p_2 \to p_1$. □

We also see on fig 10◇5 – fig 10◇6 that under the duality $C \longleftrightarrow C^\times$ selfcontacts (cusps) turns into inflections and selfintersections — to *multiple tangents*[1]. In particular, if C has at most the simple singularities, then special proper tangents of $C^\times$ are exhausted by simple inflections and double tangents.

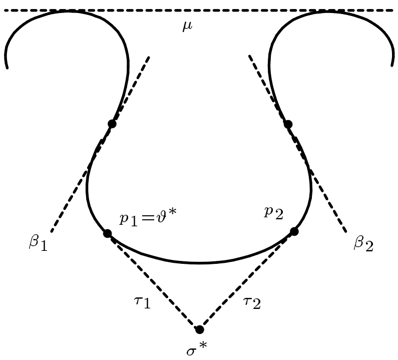


Fig. 10◇5. A curve $C \subset \mathbb{P}_2$.

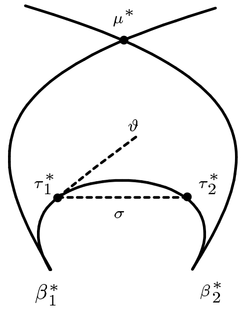


Fig. 10◇6. The dual curve $C^\times \subset \mathbb{P}_2^\times$.

10.7. Plücker identities. Let C be a curve of class c and degree d with singularities exhausted by δ ordinary selfintersections of multiplicities $m_1, m_2, \dots, m_n$ and $\varkappa$ ordinary cusps. Then

$$c = d\,(d-1) - 3\,\varkappa - \sum_{\nu=1}^{\delta} m_\nu(m_{\nu-1}) \tag{10-3}$$

If we assume, in addition, that C has only ordinary inflections, then their number $\iota = \iota(C)$ equals

$$\iota = 3\,d\,(d-2) - 8\,\varkappa - 3\sum_{\nu=1}^{\delta} m_\nu(m_{\nu-1}) \tag{10-4}$$

These formulas are known as *the Plücker identities*. We will prove them in the remaining subsections using geometric approach traced back to Chasles, Cayley and Brill[2]

Exercise 10.2. Let $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C; we write $C_q^{(d-1)}$ for $(d-1)$-th degree polar of q with respect to C. Deduce (10-3) from the equality $(C, C_q^{(d-1)}) = d(d-1)$ by proving that $(C, C_q^{(d-1)})_p$ equals 1 for smooth p, equals 3, if p is an ordinary cusp, and equals $m(m-1)$, if p is an ordinary m-typle selfintersection.

HINT. In the first case p is smooth on $C_q^{(d-1)}$ and $T_pC_q^{(d-1)} \neq T_pC$; in the second case p is smooth on $C_q^{(d-1)}$ again, but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangency; in the third case p is an $(m-1)$-typle point on $C_q^{(d-1)}$, but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$, that is, intersects it with multiplicity $(m-1)$. Now, use the Zeuthen rule.

If both C and $C^\times$ have at most the simple singularities, then the Plücker relations written for the both curves turn into

$$c = d\,(d-1) - 3\,\varkappa - 2\,\delta \qquad \iota = 3\,d\,(d-2) - 8\,\varkappa - 6\,\delta$$
$$d = c\,(c-1) - 3\,\iota - 2\,\beta \qquad \varkappa = 3\,c\,(c-2) - 8\,\iota - 6\,\beta$$

where β is a number of bitangents to C. Any three of d, c, $\varkappa$, δ, β, ι can be found from these equations as soon as the other three are known.

[1]a proper tangency is called *multiple*, if it touch the curve in several distinct points

[2]we follow the book: *J. G. Semple, L. Roth. Introduction to algebraic geometry.* (Oxford, 1949)

10.8. Blowing up $\Delta \subset \mathbb{P}_2 \times \mathbb{P}_2$. Identify $\mathbb{P}_2^\times$ with the set of lines on $\mathbb{P}_2$ and consider the incidence graph

$$\mathscr{B} \overset{\text{def}}{=} \{(p,q,\ell) \mid p,q \in \ell\} \subset \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$$

It is given by two quadratic equations $\sum \vartheta_\nu x_\nu = \sum \vartheta_\nu y_\nu = 0$ on $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$. Topologically, $\mathscr{B}$ is a 4-dimensional compact manifold. A projection $\mathscr{B} \xrightarrow{\sigma} \mathbb{P}_2 \times \mathbb{P}_2$ is bijective outside the diagonal $\Delta \overset{\text{def}}{=} \{(p,p)\} \subset \mathbb{P}_2 \times \mathbb{P}_2$. Each fiber $\sigma^{-1}(p,p) = \{(p,p,\ell) \mid \ell \ni p\}$ over $(p,p) \in \Delta$ is naturally identified with the line pencil through p on $\mathbb{P}_2$ and 3-dimensional submanifold $E \overset{\text{def}}{=} \sigma^{-1}(\Delta) \subset \mathscr{B}$ is called *an exceptional divisor*. Let us denote the projections of $\mathscr{B}$ onto consequent factors of $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$ by π_1, π_2, π_3 and write $A_1 = \pi_1^{-1}(\lambda)$, $A_2 = \pi_2^{-1}(\lambda)$, $M = \pi_3^{-1}(\lambda)$ for the full preimages of a generic line λ living on these planes. Topologically, A_1, A_2, and M are 3-dimensional cycles on $\mathscr{B}$ and their homology classes don't depend on the choice of the line λ in each plane. Any 1-parametric algebraic family of «pointed» lines $(pq) \subset \mathbb{P}_2$ can be pictured by an algebraic curve $\Gamma \subset \mathscr{B}$. Topological location of such a curve is described by a triple of numbers:

$$\begin{aligned}
\alpha_1 &= \#(\Gamma \cap A_1) && \text{— a number of } p\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\alpha_2 &= \#(\Gamma \cap A_2) && \text{— a number of } q\text{-points in } \Gamma = \{(p,q,\ell)\} \text{ laying on a generic line } \lambda \subset \mathbb{P}_2;\\
\mu &= \#(\Gamma \cap M) && \text{— a number of lines } \ell \text{ in } \Gamma = \{(p,q,\ell)\} \text{ passing through a generic point}^1\ \lambda^\times \in \mathbb{P}_2.
\end{aligned}$$

Strongly speaking, we should use the topological intersection indices instead of «the numbers of points». But for all Γ we will consider below there is an open dense set of lines[2] such that all corresponding A_1, A_2, M intersect Γ transversally in a finite constant number of points. We will always suppose that α_1, α_2, μ are calculated using A_1, A_2, M taken from these open dense sets[3]. The triples $(p,q,\ell) \in \Gamma$ with $p = q$, i.e. the intersection points $\Gamma \cap E$, are called *exceptional*. Typically, Γ has a finite number of exceptional points. Our goal is to equip the exceptional points with appropriate multiplicities and express the number $\xi(\Gamma)$ of «exceptional points counted with multiplicities» through α_1, α_2, μ in the following three examples.

10.8.1. Example: join family. Let $C_1, C_2 \subset \mathbb{P}_2$ be two curves of degrees d_1, d_2 without common components. Fix any point $u \in \mathbb{P}_2$ outside the both curves and all the lines joining pairs of their intersection points. Then

$$\Gamma = \{(p,q,\ell) \in \mathscr{B} \mid p \in C_1,\ q \in C_2,\ \ell \ni u\}$$

is a curve in $\mathscr{B}$ given by an obvious triple of algebraic equations. Its exceptional points are $(p,p,(pu))$ with $p \in C_1 \cap C_2$, that is $\xi(\Gamma)$ counts the intersections of C_1 and C_2. Further, $\alpha_1 = \alpha_2 = d_1 d_2$, because a generic line λ intersect, say C_1, in d_1 distinct points $p_1, p_2, \dots, p_{d_1}$ and for each of them $C_2 \cap (up_\nu)$ consist of d_2 points. Finally, $\mu = d_1 d_2$ too, because there is a unique line $\ell = (u\lambda^\times)$ passing through a given $\lambda^\times$ and this line contains $d_1 d_2$ distinct pairs (p_i, q_j) with $p_i \in C_1 \cap \ell$, $q_j \in C_2 \cap \ell$, if $\lambda^\times$ is general enough.

10.8.2. Example: secant family. In the above example, let $C_1 = C_2 = C$ be the same irreducible curve of degree d given by an equation $F(x) = 0$ and u be a point outside the curve, all its singular tangents, and all lines joining pairs of distinct singularities. Then a closure of $\{(p,q,\ell) \in \mathscr{B} \mid p \neq q,\ p,q \in C,\ \ell \ni u\}$ is a curve given by the equations $F(x) = F(y) = \vartheta(u) = 0$ in $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^x$. Exceptional points of this curve are $(p,p,(pu))$ such that $\mathsf{mult}(C,(pu))_p \geqslant 2$, i.e. either $(pu) = T_pC$ or $p \in \mathit{Sing}(C)$. So, ξ counts singular points of C and proper tangents coming from c to C. By the same reasons as above, $\alpha_1 = \alpha_2 = \mu = d\,(d-1)$ in this case.

10.8.3. Example: tangent family. Let C be as above and Γ be a closure of $\{(p,q,\ell) \in \mathscr{B} \mid p \neq q,\ p,q \in C,\ \ell = T_pC\}$ (it is given by the equations $F(x) = F(y) = \widetilde{F}(x^{d-1}, y) = 0$). Exceptional points of Γ are (p,p,ℓ) such that $\mathsf{mult}(C,\ell)_p \geqslant 3$, i.e. either inflection tangents at smooth p or geometric tangents at singular p. So, $\xi(\Gamma)$ counts inflections and singularities. Since a simple tangent at a smooth point p intersect the curve in $(d-2)$ more points q, we have $\alpha_1 = d\,(d-2)$. Clearly, $\mu = c\,(d-2)$, because there are c proper tangents to C through generic $\lambda^\times \in \mathbb{P}_2$ by the definition of class. Finally, $\alpha_2 = d\,(c-2)$, because a generic line intersects C in d distinct smooth points $q_1, q_2, \dots, q_d$ and for each of them there are $(c-2)$ proper tangents $(q_j p)$ touching C at some $p \neq q_j$: when a point $q \notin C$ tends to some $q_j \in C$, precisely 2 of c tangents through q turn to $T_{q_j}C$ (see fig. fig 10◇7; other arguments will appear in n° 10.10.4 and ex. 10.3).

[1] here $\lambda \subset \mathbb{P}_2^\times$ is the pencil of lines through a point $\lambda^\times \in \mathbb{P}_2$

[2] in the space of all lines

[3] it will be an exercise for readers, to check the existence of such open sets of lines in each of examples below

10.9. A correspondence on $\mathbb{P}_1$ is called *algebraic of type m-n*, if the pairs of corresponding points $(p,q)\in\mathbb{P}_1\times\mathbb{P}_1$ form an algebraic curve $\gamma\subset\mathbb{P}_1\times\mathbb{P}_1$ given by an irreducible bihomogeneous polynomial $g(t',t'')$ of bidegree (m,n) in $(t',t'')=\big((t'_0:t'_1),(t''_0:t''_1)\big)$. This curve is called *a graph* of the correspondence. So, *images* of a given point $p\in\mathbb{P}_1$ are presented by the equation $g(p,t)=0$ in $t\in\mathbb{P}_1$ and *preimages* — by the equation $g(t,p)=0$. Since these equations have respectively m and n ordinary distinct roots for almost all[1] p, a generic point has n distinct images and m distinct preimages under an m-n correspondence.

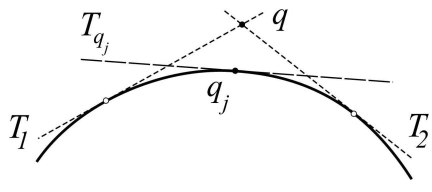


Fig 10◇7. Two proper tangent lines disappear as $q\to q_j$.

A point $e\in\mathbb{P}_1$ is called *a fixed point* of the correspondence, if it corresponds to itself[2], that is, if $g(e,e)=0$. So, the set of all fixed points is $\gamma\cap\Delta$, where $\Delta=\{(t,t)\}\subset\mathbb{P}_1\times\mathbb{P}_1$ is the diagonal. Since $g(t,t)$ is homogeneous of degree $m+n$ in $t=(t_0,t_1)$, any algebraic m-n correspondence has $m+n$ fixed points counted with multiplicities, where the multiplicity of a fixed point e means the local intersection multiplicity $(\gamma,\Delta)_{(e,e)}$, of the curve $g(t',t'')=0$ and the line $t'=t''$ at the point $(e,e)\in\gamma$. In particular, it can be calculated by the Zeuthen rule applied in any affine chart $\mathbb{A}^1\times\mathbb{A}^1=\mathbb{A}_2\ni(e,e)$.

10.9.1. CLAIM. *Let $U\subset\mathbb{P}_1$ be an affine chart with the origin at a fixed point e of a correspondence γ, x be an affine coordinate on U, and $y_1(x)$, $y_2(x)$, … , $y_m(x)$ be all γ-images of x which tend to 0 as $x\to 0$. Then $(\gamma,\Delta)_{(0,0)}$ equals the sum of orders of infinitesimals $y_i(x)-x$ with respect to x.*

Proof. Let (x,y) be affine coordinates on $\mathbb{A}^2=U\times U$ (see fig 10◇8). Since the both lines $x=0$ and $y=0$ contain just one intersection point $(0,0)\in\gamma\cap\Delta$ we can use the line pencil $x=\text{const}$ parameterized by the x-axis to calculate $(\gamma,\Delta)_{(0,0)}$ as the sum of orders of infinitesimals $\alpha_i(x)-\beta_j(x)$ where α_i and β_j run trough the intersections of a vertical line $x=\text{const}$ respectively with γ and with Δ. So, $\alpha_i(x)=y_i(x)$ and there is just one $\beta(x)=x$. $\square$

10.10. Exceptional point multiplicities. A curve $\Gamma\subset\mathscr{B}$, of pointed lines (p,q,ℓ), defines an algebraic correspondence on $\mathbb{P}_1$ as follows. Fix a point $a\in\mathbb{P}_2$ such that it lies on precisely μ distinct lines ℓ of Γ and a generic line through a contains exactly α_1 distinct p-points and exactly α_2 distinct q-points of Γ. Then consider the pencil of lines through a as $\mathbb{P}_1$ in question and say that $(ap)\longleftrightarrow(aq)$ iff $(p,q,\ell)\in\Gamma$ for some[3] ℓ. This is an algebraic α_1-α_2 correspondence, because a generic point has α_2 images and α_1 preimages, certainly. A line through a corresponds to itself under γ_Γ precisely in two cases: either it belongs to Γ, i. e. contains 2 points $p\neq q$ such that $(p,q,\ell)\in\Gamma$, or it pass through an exceptional point e such that $(e,e,\ell)\in\Gamma$ for some ℓ.

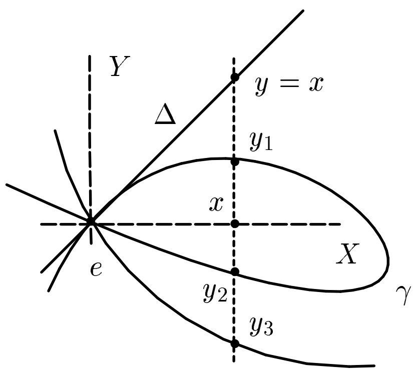


Fig. 10◇8. Fixed point.

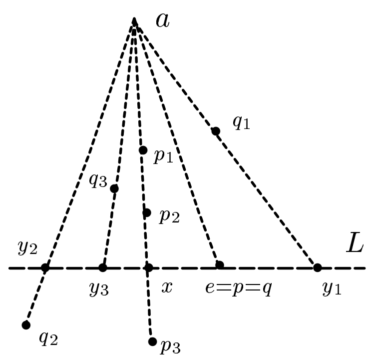


Fig. 10◇9. Exceptional point.

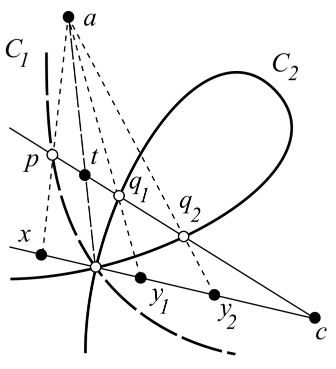


Fig. 10◇10. Intersection point.

Let us define the multiplicity of an exceptional point $(e,e,\ell)\in\Gamma$ as a multiplicity of the corresponding fixed point (ae) of the correspondence γ_Γ. By n° 10.9.1, it can be calculated geometrically as follows. Parameterize the pencil of lines through a by some line $L\not\ni a$, which pass through an exceptional point e (see fig fig 10◇9), and fix on L an affine coordinate x centered at e. Let (ay_1), (ay_2), … , (ay_m), where

[1] that is, for all p outside some finite subset on $\mathbb{P}_1$ where the both discriminants vanish
[2] of course, besides itself, a fixed point may have several other (pre) images as well
[3] of course, $\ell=(pq)$, if $p\neq q$

$y_\nu \in L$, be all lines corresponding to (ax) and tending to (ae) as $x \to e$. Then the multiplicity of (pe) equals the sum of orders of infinitesimals $y_i(x) - x$ w.r.t. x as $x \to 0$.

10.10.1. CLAIM (CHASLES–CAYLEY–BRILL FORMULA). *The total number of exceptional points counted with multiplicities equals $\xi(\Gamma) = \alpha_1 + \alpha_2 - \mu$.*

Proof. Since $\deg(\gamma_\Gamma) = (\alpha_1, \alpha_2)$, it has $\alpha_1 + \alpha_2$ fixed points. By the choice of a, each line (ap) such that $(p, q, (ap)) \in \Gamma$ for some $q \neq p$, $q \in (ap)$ has multiplicity 1 as a fixed point for γ_Γ. The residuary contribution of exceptional fixed points equals $\xi(\Gamma)$. □

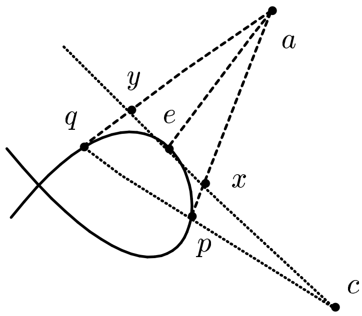


Fig. 10◇11.

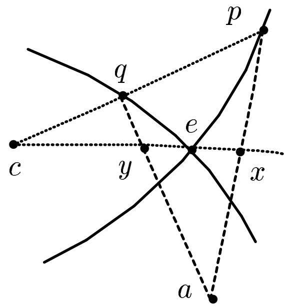


Fig. 10◇12.

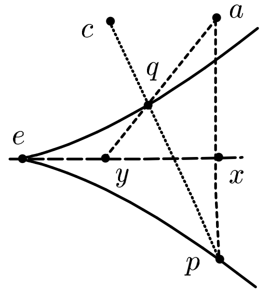


Fig. 10◇13.

10.10.2. Example: exceptional point multiplicities in join family (continuation of n° 10.8.1). In this case the multiplicity of an exceptional point $(e, e, (ce)) \in \Gamma$ coincides with the local intersection number $(C_1, C_2)_e$ (see fig. fig 10◇10). Indeed, use the line pencil centered at c to compute $(C_1, C_2)_e$ by the Zeuthen rule as it was explained in the previous lecture. If we take a outside all geometric tangents to the both curves at e and such that e is the only intersection point of C_1 and C_2 on (ae), then $(C_1, C_2)_e$ is a sum of orders $p - q_j$ w.r.t. t as $t \to 0$. But it is clear from fig. fig 10◇10 that $p - q_j$ is like $x - y_j$ and t is like x as soon the both lines (ce) and (ae) do not touch the branches of C_1, C_2 at e. So, $\xi(\Gamma)$ is the sum of local intersection numbers of C_1 and C_2, i.e. $(C_1, C_2) = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d_1 d_2 + d_1 d_2 - d_1 d_2 = d_1 d_2$. We get the Bézout theorem.

10.10.3. Example: exceptional point multiplicities in secant family (continuation of n° 10.8.2). There are 3 types of exceptional points here. A proper tangency $(e, e, (ce))$ (see fig 10◇11) has multiplicity 1, because any x closed to e has a unique image y coming to e when $x \to e$ and $y - x$ is like $x - e$. If e is an ordinary m-typle selfintersection, then a line through c closed to (ce) contains m p-points running through m branches of C; each such p-point produces $(m-1)$ q-points coming from other $(m-1)$ branches; so, there are $m(m-1)$ differences $y - x$ and each of them is like $(x - e)$ (see fig 10◇12) as soon (ce) does not touch any branch. If e is an ordinary cusp (see fig. fig 10◇13, where the line pencil through a is parameterized by the cuspidal tangent), then any x closed to e produces two p-points (intersections of (ax) with two branches of C) and each of them has just one q-point (intersection of (cp) with the other branch of C); it is easy to see from fig. fig 10◇13 that $(x - y) \sim (p - q) \sim (x - e)^{3/2}$ as $x \to e$. So, the cuspidal point contributes $2 \cdot (3/2) = 3$. Hence,

$$c + \sum_{\nu}^{\delta} m_\nu(m_\nu - 1) + 3\,\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-1) + d(d-1) - d(d-1) = d(d-1)$$

by the Chasles–Cayley–Brill formula. This gives the first Plücker identity.

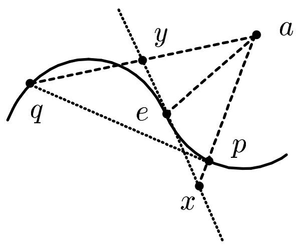


Fig. 10◇14.

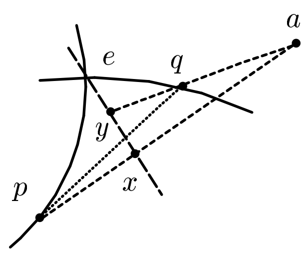


Fig. 10◇15.

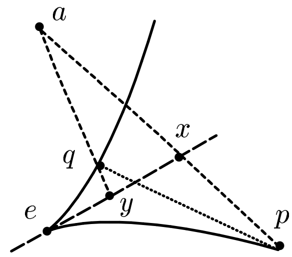


Fig. 10◇16.

10.10.4. Example: exceptional point multiplicities in tangent family (continuation of n° 10.8.3). We see on fig 10◇14 – fig 10◇16 that each difference $(y - x)$ is like $(x - e)$ for all three types of exceptional points. An inflection point (fig 10◇14) produces a unique difference and has a multiplicity 1; each of m branches through an m-typle selfintersection (fig 10◇15) produces $(m - 1)$ differences, so the multiplicity equals $m(m - 1)$ here; a cusp (fig 10◇16) produces 2 differences and has multiplicity 2. Hence,

$$\iota + \sum_{\nu}^{\delta} m_\nu(m_\nu - 1) + 2\,\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-2) + d(c-2) - c(d-2) = d^2 - 4\,d + 2\,c$$

by the Chasles–Cayley–Brill formula. Since $c = d(d-1) - \sum m_\nu(m_\nu - 1) - 3\,\varkappa$, we get the second Plücker identity $\iota = 3\,d(d-2) - 3\sum m_\nu(m_\nu - 1) - 8\,\varkappa$.

Exercise 10.3. Let c be a smooth point of C in n° 10.8.2 and n° 10.10.3. Check that a multiplicity of an exceptional point $(c, c, T_cC) \in \Gamma$ equals 2 (so, we have really $\alpha_2 = d(c-2)$ in n° 10.8.3).

§11. Affine algebraic – geometric dictionary.

In this section we assume that the ground field $\Bbbk$ is algebraically closed.

11.1. Affine varieties: their ideals and coordinate algebras. Let $X = V(J) \subset \mathbb{A}^n$ be an affine algebraic variety given by some ideal $J \subset \Bbbk[x_1, x_2, \dots, x_n]$ of polynomial equations. We write $I(X)$ for the ideal of all polynomials vanishing along X. If $\Bbbk$ is algebraically closed, then by Hilbert's Nullstellensatz

$$I(X) = \sqrt{J} \stackrel{\text{def}}{=} \{ f \in \Bbbk[x_1, x_2, \dots, x_n] \mid f^n \in J \text{ for some } n \in \mathbb{N} \}$$

is *the radical* of J. Clearly, $V(J_1) \subset V(J_2) \iff \sqrt{J_1} \supset \sqrt{J_2}$. A finitely generated commutative $\Bbbk$-algebra

$$\Bbbk[X] \stackrel{\text{def}}{=} \Bbbk[x_1, x_2, \dots, x_n]/I(X)$$

is is called *a coordinate algebra* (or *a structure algebra*) of the affine algebraic variety $X \subset \mathbb{A}^n$. Geometrically, $\Bbbk[X]$ consists of functions $X \xrightarrow{v \mapsto f(v)} \Bbbk$ obtained by restricting the polynomials $f \in \Bbbk[x_1, x_2, \dots, x_n]$ onto $X \subset \mathbb{A}^n$. These functions are called *regular* algebraic functions on X. Thus, algebra $\Bbbk[X]$ is *reduced*, i. e. has no nilpotent elements: $f^n = 0 \Rightarrow f = 0$.

Exercise 11.1. Let $X = \{O\} \in \mathbb{A}_n$ be the origin. Describe $I(X)$ and $\Bbbk[X]$.

11.1.1. PROPOSITION. *Each reduced finitely generated algebra A over an algebraically closed field can be realized as $A = \Bbbk[X]$ for some affine algebraic variety X.*

Proof. Since $A = \Bbbk[x_1, x_2, \dots, x_n]/I$ is reduced, $f^n \in I \Rightarrow f \in I$ for any $f \in \Bbbk[x_1, x_2, \dots, x_n]$. By Hilbert's Nullstellensatz, this means that $X = V(I) \subset \mathbb{A}^n$ has $I(X) = I$ and $A = \Bbbk[x_1, x_2, \dots, x_n]/I(X) = \Bbbk[X]$. $\square$

11.2. Points. Given a point $p \in X$, the evaluation map $\mathrm{ev}_p : \Bbbk[X] \xrightarrow{f \mapsto f(p)} \Bbbk$ coincides with the factorization $\Bbbk[X] \xrightarrow{f \mapsto f \pmod{\mathfrak{m}_p}} \Bbbk[X]/\mathfrak{m}_p$, where $\mathfrak{m}_p \stackrel{\text{def}}{=} \{f \in \Bbbk[X] \mid f(p) = 0\}$. Hence, $\mathfrak{m}_p = \ker(\mathrm{ev}_p)$ is a proper maximal ideal in $\Bbbk[X]$. It is called *a maximal ideal* of p.

11.2.1. PROPOSITION. *If $\Bbbk$ is algebraically closed, then the correspondences $p \longleftrightarrow \mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p$ establish bijections between the points of X, the homomorphisms $\Bbbk[X] \longrightarrow \Bbbk$ identical on $\Bbbk$, and the proper maximal ideals in $\Bbbk[X]$.*

Proof. Each $\Bbbk$-algebra homomorphism $\Bbbk[X] \longrightarrow \Bbbk$ is surjective and its kernel is a proper maximal ideal in $\Bbbk[X]$. Vice verse, for any maximal ideal $\mathfrak{m} \subset \Bbbk[X]$ the factor algebra $\Bbbk[X]/\mathfrak{m}$ is a field and is finitely generated as a $\Bbbk$-algebra. By n° 8.5.1 it is algebraic over $\Bbbk$ and hence coincides with $\Bbbk$, because $\Bbbk$ is algebraically closed. Thus, $\mathfrak{m}$ is the kernel of the canonical factorization homomorphism $\Bbbk[X] \twoheadrightarrow \Bbbk[X]/\mathfrak{m} = \Bbbk$ and the correspondence $\mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p$ is bijective.

Clearly, $p \neq q \Rightarrow \mathfrak{m}_p \neq \mathfrak{m}_q$, because we can always find a linear form $\mathbb{A}^n \xrightarrow{\varphi} \Bbbk$ such that $\varphi \in \mathfrak{m}_p$ but $\varphi \notin \mathfrak{m}_q$. To show that each proper maximal ideal $\mathfrak{m} \subset \Bbbk[X]$ is a maximal ideal of some point $p \in X$ let us write $\Bbbk[X]$ as $\Bbbk[x_1, x_2, \dots, x_n]/I(X)$. Then full preimage of $\mathfrak{m}$ is also a proper maximal ideal $\widetilde{\mathfrak{m}} \subset \Bbbk[x_1, x_2, \dots, x_n]$, because $\widetilde{\mathfrak{m}} \supset I(X)$ and $k[x_1, x_2, \dots, x_n]/\widetilde{\mathfrak{m}} = \Bbbk[X]/\mathfrak{m} = \Bbbk$. We conclude that $V(\widetilde{\mathfrak{m}}) \subset \mathbb{A}^n$ is nonempty and is contained in X. So, there is a point $p \in X$ such that $f(p) = 0$ for any $f \in \mathfrak{m}$, i. e. $\mathfrak{m} \subset \mathfrak{m}_p$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{m}_p$. $\square$

11.3. Algebraic varieties via spectra. A set of all proper maximal ideals in a given $\Bbbk$-algebra A is called *a maximal spectrum* of A and is denoted by $\mathrm{Spec}_{\mathrm{m}}(A)$. We can treat an affine algebraic variety over an algebraically closed field $\Bbbk$ pure algebraically as a maximal spectrum $\mathrm{Spec}_{\mathrm{m}}(A)$ of an arbitrary finitely generated reduced $\Bbbk$-algebra A whose elements $f \in A$ are considered as the functions $\mathrm{Spec}_{\mathrm{m}}(A) \xrightarrow{\mathfrak{m} \mapsto f \pmod{\mathfrak{m}}} \Bbbk$.

11.4. Regular morphisms of algebraic varieties. Any map of sets $X \xrightarrow{\varphi} Y$ produces *a pull back homomorphism* φ^* from the algebra $\Bbbk^Y$, of all $\Bbbk$-valued functions on Y, to the algebra $\Bbbk^X$, of all $\Bbbk$-valued functions on X. It sends $Y \xrightarrow{f} \Bbbk$ to the composition

$$\varphi^* f \stackrel{\text{def}}{=} f \circ \varphi : X \xrightarrow{\varphi} Y \xrightarrow{f} k .$$

A map $X \xrightarrow{\varphi} Y$ between affine algebraic varieties is called *a regular morphism* of algebraic varieties, if φ^* sends the regular algebraic functions on Y to the regular algebraic functions on X, i. e. induces a well defined homomorphism of coordinate algebras

$$\Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X]\,.$$

11.4.1. PROPOSITION. *Let A, B be finitely generated reduced algebras over any algebraically closed field $\Bbbk$. Then each homomorphism $B \xrightarrow{\psi} A$ such that $\psi(1) = 1$ is a pull back homomorphism $\psi = \varphi^*$ for a unique regular map $\mathrm{Spec_m}\,(A) \xrightarrow{\varphi} \mathrm{Spec_m}\,(B)$. This map φ sends a maximal ideal $\mathfrak{m} \subset A$ to its full preimage ${\varphi^*}^{-1}(\mathfrak{m}) \subset B$ and can be treated as a pull back homomorphism $\varphi = \psi^*$, for $B \xrightarrow{\psi} A$, if the points of $\mathrm{Spec_m}\,(A)$, $\mathrm{Spec_m}\,(B)$ are treated as $\Bbbk$-algebra homomorphisms $A \longrightarrow \Bbbk$, $B \longrightarrow \Bbbk$.*

Proof. Let $\mathrm{Spec_m}\,(A) \xrightarrow{\varphi} \mathrm{Spec_m}\,(B)$ be a regular morphism, $p \in \mathrm{Spec_m}\,(A)$ be a point, and $f \in B$ be a function on $\mathrm{Spec_m}\,(B)$. Then $f(\varphi(p)) = 0 \iff \varphi^* f(p) = 0$, i. e. $f \in \mathfrak{m}_{\varphi(p)} \iff \varphi^*(f) \in \mathfrak{m}_p$. So, if $B \xrightarrow{\psi} A$ is a pull back of some $\mathrm{Spec_m}\,(A) \xrightarrow{\varphi} \mathrm{Spec_m}\,(B)$, then φ has to send $\mathfrak{m}_p \longmapsto \psi^{-1}(\mathfrak{m}_p)$ for each $p \in \mathrm{Spec_m}\,(A)$. On the other hand, $\psi^{-1}(\mathfrak{m}) \subset B$ is proper maximal ideal for any proper maximal $\mathfrak{m} \subset A$, because $\varphi^*(1) = 1 \Rightarrow B/\psi^{-1}(\mathfrak{m}) = \mathrm{im}\,(\psi)/(\mathfrak{m} \cap \mathrm{im}\,(\psi)) \simeq \Bbbk$. So, $\varphi : \mathrm{Spec_m}\,(A) \xrightarrow{\mathfrak{m}_p \mapsto \psi^{-1}(\mathfrak{m}_p)} \mathrm{Spec_m}\,(B)$ is well defined map of sets. To compute its pull back homomorphism, note that for any $b \in B$, $p \in \mathrm{Spec_m}\,(A)$ we have

$$\varphi^* b(p) = b(\varphi(p)) = b\ \big(\mathrm{mod}\ \mathfrak{m}_{\varphi(p)}\big) = b\ \big(\mathrm{mod}\ \psi^{-1}(\mathfrak{m}_p)\big) = \psi(b)\ \left(\mathrm{mod}\ \mathfrak{m}_p\right) = \psi b(p)\,.$$

So, $\varphi^* = \psi$ as required. □

11.5. Geometric schemes. Let A be an arbitrary algebra over a field $\Bbbk$. All nilpotent elements of A form an ideal $\mathfrak{n}(A) \subset A$. It is contained in any proper maximal ideal $\mathfrak{m} \subset A$, because $A/\mathfrak{m} = \Bbbk$ has no nilpotent elements. So, a factor algebra $A_{\mathrm{red}} \overset{\mathrm{def}}{=} A/\mathfrak{n}$ is reduced and has the same maximal spectrum $X = \mathrm{Spec_m}\, A = \mathrm{Spec_m}\, A_{\mathrm{red}}$. If $\Bbbk$ is algebraically closed, then the intersection of all maximal ideals in A_{red} is zero, because it consists of all functions vanishing everywhere on the affine algebraic variety X. Hence, the intersection of all maximal ideals in A coincides with $\mathfrak{n}(A)$.

Exercise 11.2. Show that in general situation, when A is an arbitrary commutative ring, $\mathfrak{n}(A)$ coincides with the intersection of all proper *prime*[1] ideals $\mathfrak{p} \subset A$

A pair $(A, \mathrm{Spec_m}\, A)$, where A is an arbitrary finitely generated algebra over algebraically closed field, is called *an affine geometrical scheme*. Affine algebraic variety $X = \mathrm{Spec_m}\, A = \mathrm{Spec_m}\, A_{\mathrm{red}}$ is called *a support* of this scheme. Intuitively, the scheme differs from X by allowing some «infinitezimals», i. e. nilpotent «functions» whose «numerical values« vanish everywhere on X. Usually, these nilpotents encode some «multiplicities» attached to X.

11.5.1. Example: an intersection of affine algebraic varieties $X, Y \subset \mathbb{A}^n$ is defined as $V(I(X) + I(Y))$, i. e. by the union of all equations for X and Y. If the intersection is non-transversal, a factor algebra

$$A = k[x_1, x_2, \dots, x_n]/(I(X) + I(Y))$$

is not reduced. Say, if $I(X) = (x)$, $I(Y) = (x^2 - y)$ in $\Bbbk[x, y]$, then the factor algebra $A = \Bbbk[x, y]/(x, y^2 - x) \simeq \Bbbk[y]/(y^2)$ has quadratic nilpotent y. Geometrically, the intersection of the line $X = V(x)$ and the parabola $Y = V(x^2 - y)$ consists of unique point $\mathrm{Spec_m}\,(\Bbbk) = \mathrm{Spec_m}\, A_{\mathrm{red}}$, where they touch each other with multiplicity 2. This multiplicity can be extracted from non reduced algebra A but it is lost under the replacement of A by A_{red}. Thus, if we want, say, to develop an intersection multiplicities technique, then we have to treat intersections as *geometric schemes* rather than algebraic varieties and investigate the difference between A and A_{red}.

By the definition, a *regular morphism* $(A, \mathrm{Spec_m}\, A) \xrightarrow{(\varphi^*, \varphi)} (B, \mathrm{Spec_m}\, B)$ of schemes is a pair that consists of an algebra homomorphism $B \xrightarrow{\varphi^*} A$ and a regular map $\mathrm{Spec_m}\,(A) \xrightarrow{\varphi} \mathrm{Spec_m}\,(B)$ such that $\varphi^* f(p) = f(\varphi(p))$ for all $f \in B$, $p \in \mathrm{Spec_m}\,(A)$. Note that now φ^* can not be recovered from φ, because the latter one knows nothing about the nilpotent infinitesimals.

[1] an ideal $\mathfrak{p} \subset A$ is called *prime*, if $A/\mathfrak{p}$ has no zero divisors

11.6. A direct product of affine algebraic varieties. Let A, B be finitely generated $\Bbbk$-algebras. Then the tensor product of algebras[1] $A\otimes B$ is a finitely generated $\Bbbk$-algebra with the multiplication

$$(A\otimes B)\times(A\otimes B)\xrightarrow{(a\otimes b,\alpha\otimes\beta)\mapsto a\alpha\otimes b\beta} A\otimes B\,.$$

11.6.1. PROPOSITION. *A set-theoretical product $X\times Y$, of affine algebraic varieties $X=\operatorname{Spec_m} A$, $Y=\operatorname{Spec_m} B$, is naturally identified with $\operatorname{Spec_m}(A\otimes B)$. Both projections*

$$X\xleftarrow{\pi_X} X\times Y\xrightarrow{\pi_Y} Y$$

are regular morphisms w.r.t. the structure of an affine algebraic variety on $X\times Y$ prescribed by this identification and for any two regular maps $X\xleftarrow{\varphi} Z\xrightarrow{\psi} Y$ there exists a unique regular map $Z\xrightarrow{\varphi\times\psi} X\times Y$ that fits into commutative diagram

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \\ & Z \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] \arrow[uu, "\varphi\times\psi"] & \end{tikzcd}$$

Proof. Assume for a moment that $A\otimes B$ is reduced and define $X\times Y$ as $\operatorname{Spec_m}(A\otimes B)$ and the projections π_X, π_Y as the regular maps whose pull-back homomorphisms are the canonical algebra inclusions

$$\alpha\,:\,A\xhookrightarrow{a\mapsto a\otimes 1} A\otimes B\xleftarrow{1\otimes b\leftarrow b} B\,:\,\beta\,.$$

Then the last asseveration of the proposition turns to the universal property which characterizes the tensor product of algebras. Namely, if $Z=\operatorname{Spec_m}(C)$, then for any two algebra homomorphisms $A\xrightarrow{\varphi^*} C\xleftarrow{\psi^*} B$ there is a unique homomorphism $A\otimes B\xrightarrow{\varphi^*\otimes\psi^*} C$ such that $(\varphi^*\otimes\psi^*)\circ\alpha=\varphi^*$ and $(\varphi^*\otimes\psi^*)\circ\beta=\psi^*$.

Exercise 11.3. Deduce this property from the universality of the tensor product of vector spaces discussed in §4. In particular, for the points of Z, i.e. the regular morphisms $\operatorname{Spec_m}(\Bbbk)\longrightarrow Z$ (or, equivalently, the algebra homomorphisms $C\twoheadrightarrow\Bbbk$)), we get a set-theoretical bijections

$$\operatorname{Spec_m}(A)\times\operatorname{Spec_m}(B)\simeq\operatorname{Hom}(\operatorname{Spec_m}(\Bbbk),X)\times\operatorname{Hom}(\operatorname{Spec_m}(\Bbbk),Y)\simeq$$
$$\simeq\operatorname{Hom}(\operatorname{Spec_m}(\Bbbk),X\times Y)\simeq\operatorname{Spec_m}(A\otimes B)\,.$$

Thus, it remains to show that $A\otimes B$ is reduced. We can write $f\in A\otimes B$ as $\sum a_\nu\otimes b_\nu$, where the functions $b_\nu\in B$ are linearly independent over $\Bbbk$. If f produces the identically zero function on $X\times Y$, i.e. $f(p,q)=0$ $\forall(p,q)\in X\times Y$, then $\sum a_\nu(p)\cdot b_\nu=0$ in $\Bbbk[Y]$. Hence, all $a_\nu(p)=0\quad\forall p\in X$. Thus, all $a_\nu=0$ in A and $f=0$ in $A\otimes B$. □

11.7. Zariski topology. Any affine algebraic variety $X=\operatorname{Spec_m} A$ admits a canonical topology whose closed sets are $V(I)=\{x\in X\mid f(x)=0\quad\forall f\in I\}$, where $I\subset A$ is an arbitrary ideal. This topology is called *the Zariski topology*.

Exercise 11.4. Check that $V(I)$ satisfy the closed set properties, namely: $\varnothing=V(1)$; $X=V(0)$; $\bigcap V(I_\nu)=V\left(\sum I_\nu\right)$, where $\sum I_\nu$ consists of all finite sums $\sum f_\nu$ with $f_\nu\in I_\nu$; $V(I)\cup V(J)=V(IJ)$, where IJ is an ideal spanned by all products ab with $a\in I$, $b\in J$.

Since any ideal $I\subset\Bbbk[X]$ is finitely generated, each closed set is a finite intersection of hypersurfaces: $V(I)=V(f_1,f_2,\dots,f_m)=\bigcap V(f_\nu)$. Hence, any Zariski open set is a finite union of *principal* open sets $\mathscr{D}(f)\overset{\text{def}}{=}X\setminus V(f)=\{x\in X\mid f(x)\neq 0\}$.

[1]as a vector space over $\Bbbk$ it coincides with the tensor product of vector spaces A, B and consists of all *finite* sums $\sum a_\nu\otimes b_\nu$ with $a_\nu\in A$, $b_\nu\in B$; for example, $\Bbbk[x]\otimes\Bbbk[y]$ is naturally isomorphic to $\Bbbk[x,y]$

The Zariski topology has a pure algebraic nature. Since the Zariski neighborhoods express rather some divisibility conditions than any distance relations, their properties are far enough from the metric topology standards.

11.7.1. Example: irreducible closed sets. A topological space X is called *reducible*, if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subset X$. This is a vapid notion in the usual metric topology, where everything is reducible. In Zariski topology, the reducibility of X means an existence of non zero functions $f_1, f_2 \in \Bbbk[X]$ such that f_1 vanishes along X_1 and f_2 vanishes along X_2. Since $f_1 f_2$ vanishes everywhere, $f_1 f_2 = 0$ in $\Bbbk[X]$. So, X is reducible iff $\Bbbk[X]$ has zero divisors. For example, a hypersurface $\{g(x) = 0\} \subset \mathbb{A}^n$ is irreducible iff g is a power of an irreducible polynomial. Irreducible algebraic sets are similar to the prime numbers in arithmetics.

11.7.2. PROPOSITION. *Any affine algebraic variety admits a unique finite decomposition $X = \bigcup X_i$, where $X_i \subset X$ are irreducible proper closed subsets such that $X_i \not\subset X_j\ \forall\, i \neq j$ (they are called irreducible components of X).*

Proof. A decomposition is constructed step by step. If X is reducible, the first step takes $X = Z_1 \cup Z_2$, where $Z_{1,2}$ are proper closed subsets. Let $X = \bigcup Z_\nu$ after n steps. If each Z_ν is irreducible, then for each ν and any irreducible closed subset $Y \subset X$ either $Y \cap Z_\nu = \varnothing$ or $Y \subset Z_\nu$, because of $Y = \bigcup (Z_\nu \cap Y)$. So, if we take away all Z_ν contained in some other Z_μ, then we get the required decomposition and it is unique. If there are some reducible Z_ν after n steps, $(n+1)$-th step replaces each of them by a union of two proper closed subsets. If this procedure would never stop, then it produces an infinite chain of strictly embedded subsets $X \supset Y_1 \supset Y_2 \supset \dots$, i. e. an infinite chain of strictly increasing ideals $(0) \subset I_1 \subset I_2 \subset \dots$, which does not exist in the Noetherian algebra $\Bbbk[X]$. □

11.7.3. Example: «big» open sets. Zariski topology is week and non Hausdorf. For example, $Z \subset \mathbb{A}^1$ is Zariski closed iff Z is finite. If X is irreducible, then any non empty open $U_1, U_2 \subset X$ have a nonempty intersection, because in the contrary case $X = (X \setminus U_1) \cup (X \setminus U_2)$.

Exercise 11.5. Prove that $f = g$ in $\Bbbk[X]$, if X is irreducible and $f|_U = g|_U$ over some open non-empty $U \subset X$.
HINT. If $U = \mathscr{D}(h)$, then $X = V(h) \cup V(f - g)$.

11.7.4. Example: Zariski topology on $X \times Y$ is finer than the product of Zariski topologies on X and Y, because the closed $Z \subset X \times Y$ are not exhausted by the products of closed subsets on X, Y. For example, if $X = Y = \mathbb{A}^1$, then any curve, say a hyperbola $V(xy - 1)$, is closed in Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed sets on $\mathbb{A}^1$ are exhausted by finite unions of isolated points and coordinate lines.

11.7.5. PROPOSITION. *A regular morphism $X \xrightarrow{\varphi} Y$ of algebraic varieties is continuous in Zariski topology.*

Proof. A preimage $\varphi^{-1}(Z)$ of a closed $Z = V(I) \subset Y$ consists of all $x \in X$ such that $0 = f(\varphi(x)) = {\varphi^*}^{-1} f(x)$ for all $f \in I$. So, it coincides with the zero set of an ideal ${\varphi^*}^{-1}(I) \subset \Bbbk[X]$. □

11.8. Decomposition of regular morphism. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of affine algebraic varieties. Then, $\Bbbk$-algebra homomorphism $\Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X]$ can be decomposed as

$$\Bbbk[Y] \twoheadrightarrow \mathrm{im}\,(\varphi^*) \hookrightarrow \Bbbk[X]\,.$$

Since $\Bbbk[X]$ is reduced, $\mathrm{im}\,(\varphi^*) \subset \Bbbk[X]$ is reduced too, i. e. there is an affine algebraic variety

$$Z = \mathrm{Spec}_{\mathrm{m}}\, \mathrm{im}\,(\varphi^*)$$

such that $X \xrightarrow{\varphi} Y$ is decomposed as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} Y$ and $\Bbbk[Y] \xrightarrow{\varphi_2^*} \Bbbk[Z]$ is surjective, $\Bbbk[Z] \xrightarrow{\varphi_1^*} \Bbbk[X]$ is injective. The injectivity of φ_1^* means that non zero function $f \in \Bbbk[Z]$ can not vanish along $\varphi_1(X)$, i. e. $\varphi_1(X) \subset Z$ is dense. The surjectivity of φ_2^* means that φ_2 induces an isomorphism between Z and a closed subset $V(\ker \varphi_2^*) \subset Y$ given by the ideal[1] $\ker(\varphi_2^*) = \ker(\varphi) \subset \Bbbk[Y]$. In other words, the Zariski closure $Z = \overline{\varphi(X)} \subset Y$ is an affine algebraic variety and the maps $X \longrightarrow \overline{\varphi(X)} \hookrightarrow Y$ are regular morphisms.

[1] note that this ideal is automatically radical

11.9. Dominant morphisms, closed embeddings, and finite morphisms. If X is irreducible and a homomorphism $\Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X]$ is injective, then the corresponding morphism $X \xrightarrow{\varphi} Y$ is called *dominant*. Geometrically, this means that $\overline{\varphi(X)} = Y$. If X is reducible, then φ is called *dominant* when its restriction onto *each* irreducible component of X is dominant.

A morphism $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $\Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X]$ is surjective. This means that φ identifies X with $V(\ker\varphi^*) \subset Y$.

Exercise 11.6. Show that any dominant morphism of irreducible affine varieties $X \xrightarrow{\varphi} Y$ can be decomposed as

$$X \xhookrightarrow{\psi} Y \times \mathbb{A}^m \xrightarrow{\pi} Y \;, \tag{11-1}$$

where ψ is a closed embedding and π is the natural projection along $\mathbb{A}^m$.

HINT. Let $A = \Bbbk[X]$, $B = \Bbbk[Y]$. A has a natural structure of a finitely generated B-algebra provided by the inclusion $B \xhookrightarrow{\varphi^*} A$; thus, there is an epimorphism of B-algebras $B[x_1, x_2, \dots, x_m] \xrightarrow{\psi} A$ for some m.

Given a regular morphism $X \xrightarrow{\varphi} Y$, then the coordinate algebra $\Bbbk[X]$ can be considered as an algebra over $\varphi^*(\Bbbk[Y]) = \Bbbk[\overline{\varphi(X)}] \subset \Bbbk[X]$. A morphism φ is called *finite*, if $\Bbbk[X]$ is integer over $\varphi^*(\Bbbk[Y])$. Since $\Bbbk[X]$ is finitely generated as algebra over $\varphi^*(\Bbbk[Y])$ (even over $\Bbbk$), finiteness of φ means that $\Bbbk[X]$ is finitely generated *as* $\varphi^*(\Bbbk[Y])$*- module*, i. e. there are some $f_1, f_2, \dots, f_m \in \Bbbk[X]$ such that any $h \in \Bbbk[X]$ can be written as $h = \sum \varphi^*(g_i)\, f_i$ for some $g_i \in \Bbbk[Y]$.

11.9.1. PROPOSITION. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of affine algebraic varieties. Then $\varphi(Z)$ is closed for any closed $Z \subset X$ and induced morphism $Z \xrightarrow{\varphi|_Z} \varphi(Z)$ is finite. Moreover, if X is irreducible and $Z \neq X$, then $\varphi(Z) \neq Y$.*

Proof. Let $I = I(Z) \subset \Bbbk[X]$ be the ideal of $Z \subset X$. Then $Z \xrightarrow{\varphi|_Z} Y$ has $\varphi_Z^* : \Bbbk[Y] \xrightarrow{\varphi^*} \Bbbk[X] \longrightarrow \Bbbk[X]/I$. Since $\Bbbk[X]$ is finitely generated as $\varphi^*(\Bbbk[Y])$-module, $\Bbbk[Z] = \Bbbk[X]/I$ is finitely generated as a module over

$$\Bbbk[\,\overline{\varphi(Z)}\,] = \varphi|_Z^*(\Bbbk[Y]) = \varphi^*(\Bbbk[Y])/(I \cap \varphi^*(\Bbbk[Y])) \;,$$

i. e. $Z \longrightarrow \overline{\varphi(Z)}$ is a finite morphism. To prove that $\varphi(Z) = \overline{\varphi(Z)}$, we can restrict ourself onto irreducible components of Z, i. e. suppose that Z is irreducible. Let $B = \Bbbk[Z]$, $A = \Bbbk[\,\overline{\varphi(Z)}\,] \subset B$, and $f_1, f_2, \dots, f_m$ generate B as A-module. Since $\varphi|_Z$ takes a maximal ideal $\mathfrak{m}_p \subset B$ to the maximal ideal $\mathfrak{m}_p \cap A \subset A$, a point $q \in \operatorname{Spec_m}(A)$ does not belong to $\operatorname{Spec_m}(B)$ iff its maximal ideal $\mathfrak{m}_q \subset A$ generates non proper ideal in B, that is $\mathfrak{m}_q \cdot B = B$. In this case we can write $f_i = \sum \beta_{i\nu} f_\nu$ for some $\beta_{i\nu} \in \mathfrak{m}_q$; that is the zero homomorphism of A-modules: $B \longrightarrow 0$, which takes each f_i to zero, can be presented in terms of the generator system $\{f_\nu\}$ by the matrix $E - (\beta_{\nu i})$. Hence, the multiplication by $\det(E - (\beta_{ij}))$ annihilates B. Since there no zero divisors in $B = \Bbbk[Z]$, we get $\det(E - (\beta_{ij})) = 0$. But $\det(E - (\beta_{ij})) = 1 + \beta$ where $\beta \in \mathfrak{m}_q$. So, $1 \in \mathfrak{m}_q$ and $\mathfrak{m}_q \subset A$ is non proper as well.

To prove that $\varphi(Z) \neq Y$ for $Z \subsetneq X$, let us take a non zero function $f \in \Bbbk[X]$ vanishing along Z and write the integer equation of the lowest possible degree for f over $\varphi^*(\Bbbk[Y])$ as

$$f^m + \varphi^*(g_1)\, f^{m-1} + \cdots + \varphi^*(g_{m-1})\, f + \varphi^*(g_m) = 0 \;.$$

Computing its left side at any $z \in Z$, we get $\varphi^*(g_m)(z) = 0$, that is $g_m(\varphi(z)) = 0$. So, if $\varphi(Z) = Y$, then $g_m \equiv 0$ along Y, i. e. $\varphi^*(g_m) = 0$ in $\Bbbk[X]$. Since $\Bbbk[X]$ has no zero divisors, the minimal equation above is divisible by f. Contradiction. □

11.10. Normal algebraic varieties. If Y is irreducible, then $\Bbbk[Y]$ has no zero divisors. Its quotient field is called *the field of rational functions* on Y and is denoted $\Bbbk(Y)$. An irreducible variety Y is called *normal*, if $\Bbbk[Y]$ is a normal ring, i. e. there are no rational functions $f \in \Bbbk(Y) \setminus \Bbbk[Y]$ integer over $\Bbbk[Y]$. By n° 8.7.1, any algebraic variety X with factorial coordinate algebra $\Bbbk[X]$ is normal. For example, all affine spaces $\mathbb{A}^n$ are normal.

11.10.1. PROPOSITION. *Let $X \xrightarrow{\varphi} Y$ be a surjective finite morphism. If Y is normal, then $\varphi(U)$ is open for any open $U \subset X$ and each irreducible component of X is surjectively mapped onto Y.*

Proof. We will identify $\Bbbk[Y]$ with a subalgebra of $\Bbbk[X]$ embedded into $\Bbbk[X]$ via φ^*. To prove the first assertion, we can suppose that $U = \mathscr{D}(f)$ is principal. Then for any $p \in \mathscr{D}(f)$ it is enough to find $a \in \Bbbk[Y]$ such that $\varphi(p) \in \mathscr{D}(a) \subset \varphi(\mathscr{D}(f))$. Consider a map

$$\psi = \varphi \times f \;:\; X \xrightarrow{p \mapsto (\varphi(p), f(p))} Y \times \mathbb{A}^1 \;. \tag{11-2}$$

It is regular and finite, because its pull back homomorphism is the evaluation map

$$\psi^* : \Bbbk[Y \times \mathbb{A}^1] = \Bbbk[Y][t] \xrightarrow{t \mapsto f} \Bbbk[X] \tag{11-3}$$

and $\Bbbk[X]$ is finitely generated as $\Bbbk[Y]$-module. We can treat evaluation (11-3) as taking values in a $\Bbbk(Y)$-algebra $B = \Bbbk(Y) \underset{\Bbbk[Y]}{\otimes} \Bbbk[X]$, which consists of all fractions b/a, where $b \in \Bbbk[X]$, $a \in \Bbbk[Y]$, $a \neq 0$, modulo the equivalence

$$b'/a' \sim b''/a'' \quad \iff \quad b''a' - b'a'' \text{ divides zero in } \Bbbk[X]$$

($\Bbbk[X]$ is mapped into B via $f \longmapsto f/1$). Since f is integer over $\Bbbk[Y]$, f is algebraic over $\Bbbk(Y)$ and the kernel of the extended evaluation $\Bbbk(Y)[t] \xrightarrow{t \mapsto f} B$ is a principal ideal $(\mu_f) \subset \Bbbk(Y)[t]$ spanned by the minimal polynomial $\mu_f(t) = t^m + a_1 t^{m-1} + \cdots + a_{m-1} t + a_m \in \Bbbk(Y)[t]$ for f over $\Bbbk(Y)$. By n° 8.4.2, the the coefficients of μ_f belong to $\Bbbk[Y]$, i. e. $\mu_f \in \Bbbk[Y \times \mathbb{A}^1]$. Thus, $\ker \psi^* = (\mu_f)$ and $\overline{\operatorname{im} \psi} = \operatorname{Spec}_{\mathrm{m}} \left(\Bbbk[Y \times \mathbb{A}^1]/(\mu_f)\right) = V(\mu_f)$.

In other words, regular morphism (11-2) gives a finite surjection of X onto hypersurface $V(\mu_f) \subset Y \times \mathbb{A}^1$ and the initial morphism φ is obtained from (11-2) by projecting this hypersurface onto Y. Thus, for any $y \in Y$ the set $\varphi^{-1}(y) \subset X$ is surjectively mapped by f onto the set of all roots of the polynomial

$$\mu_f(y;t) = t^m + a_1(y)\, t^{m-1} + \cdots + a_m(y) \in \Bbbk[t]$$

obtained by evaluating the coefficients of μ_f at $y \in Y$. In particular, $\mathscr{D}(f) \cap \varphi^{-1}(y)$ is sent by f to non-zero roots. We conclude that $y \in \varphi(\mathscr{D}(f)$ iff $\mu_f(y;t) \in \Bbbk[t]$ has a non zero root (i. e. $\mu_f(y;t) \neq t^m$). Since $p \in \mathscr{D}(f)$, the polynomial $\mu_f(\varphi(p);t)$ should have some non zero intermediate coefficient $a_i(\varphi(p)) \neq 0$, $i < m$. This forces $\mu_f(y;t)$ to have a non zero root over each $y \in \mathscr{D}(a_i) \subset Y$. We conclude that $\varphi(\mathscr{D}(f)) \supset \mathscr{D}(a_i) \ni \varphi(p)$ as required.

What about irreducible components, consider the irreducible decomposition $X = \cup X_\nu$. Then

$$U_i = X \setminus \bigcup_{\nu \neq i} X_\nu = X_i \setminus \bigcup_{\nu \neq i} (X_i \cap X_\nu)$$

is open in X and dense in X_i. Since $\varphi(U_i)$ is non empty open subset of Y, $\varphi(X_i) = \overline{\varphi(U_i)} = Y$. $\square$

§12. Algebraic manifolds.

12.1. Localization. Let $U \subset X$ be an open subset of an affine algebraic variety and $u \in U$. A function $U \xrightarrow{f} \Bbbk$ is called *regular* at u, if there are $p, q \in \Bbbk[X]$ such that $q(u) \neq 0$ and $f(x) = p(x)/q(x)$ $\forall x \in \mathscr{D}(q) \cap U$. All functions $U \xrightarrow{f} \Bbbk$ regular at any $u \in U$ form a commutative ring denoted by $\mathscr{O}_X(U)$ or by $\Gamma(U, \mathscr{O}_X)$. It is called a ring of *local regular functions* on $U \subset X$.

12.1.1. CLAIM. *Let X be irreducible and $h \in \Bbbk[X]$. Then any $f \in \mathscr{O}_X(\mathscr{D}(h))$ can be written as $f(x) = r(x)/h^d(x)$ for appropriate $r \in \Bbbk[X]$, $d \in \mathbb{N}$. In particular, for $h \equiv 1$, we get $\mathscr{O}_X(X) = \Bbbk[X]$.*

Proof. If $f \in \mathscr{O}_X(\mathscr{D}(h))$, then $\forall u \in \mathscr{D}(h)$ there are $p_u, q_u \in \Bbbk[X]$ such that $q_u(u) \neq 0$ and $f(x) = p_u(x)/q_u(x)$ for all $x \in \mathscr{D}(q_u) \cap \mathscr{D}(h)$. So, $\bigcap_{u \in U} V(q_u)$ sits inside $V(h)$ and, by Hilbert's Nullstellensatz, some power h^d belongs to the ideal spanned by q_u, i.e. there are some $u_1, u_2, \dots, u_m \in \mathscr{D}(h)$ such that $h^d = \sum q_{u_\nu} g_\nu$ for appropriate $g_1, g_2, \dots, g_m \in \Bbbk[X]$. At the same time, $f(x)\, q_{u_\nu}(x) = p_{u_\nu}(x)$ for each ν and any $x \in \mathscr{D}(h)$, including $x \in \mathscr{D}(h) \cap V(q_{u_\nu})$. Indeed, let $q_{u_\nu}(w) = 0$ for some $w \in \mathscr{D}(h)$. Rewriting $f = p_{u_\nu}/q_{u_\nu}$ as p_w/q_w with $q_w(w) \neq 0$, we get $p_{u_\nu}(x)\, q_w(x) = q_{u_\nu}(x)\, p_w(x)$ for all $x \in \mathscr{D}(h \cdot q_{u_\nu} \cdot q_w)$. By ex. 11.5, this holds for any $x \in X$ at all. In particular, $p_{u_\nu}(w) = q_{u_\nu}(w)\, p_w(w)/q_w(w) = 0$. We conclude that $f\, h^d = \sum f\, q_{u_\nu}\, g_\nu = \sum p_{u_\nu}\, g_\nu \in \Bbbk[X]$. □

12.1.2. COROLLARY. *Any principal open set $\mathscr{D}(f) = \mathrm{Spec_m}\, \Bbbk[X][f^{-1}]$ is an affine algebraic variety, the inclusion $\mathscr{D}(f) \hookrightarrow X$ is a regular map with the pull back homomorphism $\Bbbk[X] \hookrightarrow \Bbbk[X][f^{-1}]$.* □

12.2. Structure sheaf. The correspondence $\mathscr{O}_X : U \longmapsto \mathscr{O}_X(U)$ is called *a structure sheaf* of an *affine algebraic manifold* X. If $U = \bigcup W_i$ is an union of open sets, then $U \xrightarrow{f} \Bbbk$ is regular iff each its restriction $f|_{W_i}$ is regular on W_i. Conversely, a collection of functions $W_i \xrightarrow{f_i} \Bbbk$ such that $f_i \equiv f_j$ on $W_i \cap W_j$ gives a unique regular function $f \in \mathscr{O}_X(\cup W_i)$ whose restriction onto W_i is f_i for all i.

Note that although n° 12.1.2 says that open sets are *locally* affine, a generic open $U \subset X$ is *not* an affine algebraic variety and in general there is no natural 1–1 correspondence between the points of $\mathrm{Spec_m}\, \mathscr{O}_X(U)$ and the ones of U.

Exercise 12.1. Let $U = \mathbb{A}^n \setminus O$ be the complement to the origin. Show that $\mathscr{O}_{\mathbb{A}^n}(U) = \Bbbk[\mathbb{A}^n]$ for $n \geqslant 2$.
HINT. Use the covering $U = \bigcup \mathscr{D}(x_i)$ and n° 12.1.1.

12.3. Algebraic manifolds. Let X be a topological space. An open subset $U \subset X$ is called *an algebraic affine chart* on X, if there exists an affine algebraic variety X_U and *a homeomorphism* $X_U \xrightarrow{\varphi_U} U$. Two algebraic charts $X_U \xrightarrow{\varphi_U} U$ and $X_W \xrightarrow{\varphi_W} W$ on X are called *compatible*, if their *transition map* $\varphi_{WU} = \varphi_W \circ \varphi_U^{-1}$, which identifies $\varphi_U^{-1}(U \cap W) \subset X_U$ with $\varphi_W^{-1}(U \cap W) \subset X_W$, is a regular isomorphism of algebraic open sets, i.e. its pull back $\Gamma\left(\varphi_W^{-1}(U \cap W),\, \mathscr{O}_{X_W}\right) \xrightarrow{\varphi_{WU}^*} \Gamma\left(\varphi_U^{-1}(U \cap W),\, \mathscr{O}_{X_U}\right)$ is a well defined isomorphism of $\Bbbk$-algebras. A (finite) open covering $X = \bigcup U_\nu$ by mutually compatible algebraic charts is called a (finite) *algebraic atlas* on X. Two algebraic atlases are called *equivalent*, if their union is an algebraic atlas too. A topological space X equipped with a class of equivalent (finite) algebraic atlases is called *an algebraic manifold* (of finite type).

Exercise 12.2. Check that projective spaces and Grassmannians are algebraic manifolds of finite type as well as any zero set of a collection of multihomogeneous polynomials on $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$.

12.3.1. Example: a direct product $X \times Y$ of algebraic manifolds X, Y is an algebraic manifold too. Its atlas consists of all pairwise products $U \times W$, where $U \subset X$, $W \subset X$ are affine algebraic charts on X, Y.

12.3.2. Example: separability. The standard atlas on $\mathbb{P}_1$ consists of two charts $\varphi_i : \mathbb{A}_1 \xrightarrow{\sim} U_i \subset \mathbb{P}_1$, $i = 0, 1$, and $\varphi_0^{-1}(U_0 \cap U_1) = \varphi_1^{-1}(U_0 \cap U_1) = \{t \in \mathbb{A}^1 \mid t \neq 0\}$ is the complement to the origin. The charts $U_{0,1} = \mathbb{A}^1$ are glued together along $\mathbb{A}^1 \setminus \{O\}$ via transition map $\varphi_{01} : t \mapsto 1/t$. If we replace it by the identity map $\widetilde{\varphi}_{01} : t \mapsto t$, then we get an other manifold called «$\mathbb{A}^1$ with doubled origin», which looks like ——:——. Such the pathology is known as a *non-separability*. It has appeared, because the latter gluing rule $\widetilde{\varphi}_{01}$ is «non-complete» and could be extended from $\mathbb{A}^1 \setminus \{O\}$ to a larger set. This can be formalized as follows. Two inclusions $U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1$ give an embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1$. In the case of $\mathbb{P}_1$, this is an inclusion $(\mathbb{A}^1 \setminus O) \hookrightarrow \mathbb{A}^2$ given as $x = t\,;\ y = 1/t$; it identifies $U_0 \cap U_1$ with *a closed* subset $V(xy - 1) \subset \mathbb{A}^2 = U_0 \times U_1$. In the second case, the embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1 = \mathbb{A}^2$ is given as $x = t\,;\ y = t$ and has a *non-closed* image

$\Delta \setminus \{(0,0)\}$, where $\Delta = V(x-y) \subset \mathbb{A}^2$ is the diagonal. An algebraic manifold X is called *separable*, if an image of the canonical embedding $U \cap W \hookrightarrow U \times W$ is closed for any two affine charts $U, W \subset X$. Since this image is nothing more than the intersection of diagonal $\Delta \subset X \times X$ with the affine chart $U \times W$ on $X \times X$, a manifold X is separable iff the diagonal $\Delta \subset X \times X$ is closed. For example, $\mathbb{A}^n$ and $\mathbb{P}_n$ are separable, because the diagonals in $\mathbb{A}^n \times \mathbb{A}^n$ and in $\mathbb{P}_n \times \mathbb{P}_n$ are given, accordingly, by the equations $x_i = y_i$ and $x_i y_j = x_j y_i$.

12.4. Regular functions and morphisms. Let $U \subset X$ be an open set. A function $U \xrightarrow{f} \Bbbk$ is called *regular*, if each point $u \in U$ has an affine neighborhood $X_W \xrightarrow{\varphi_W} W \ni u$ such that $f \circ \varphi \in \mathscr{O}_{X_W}\left(\varphi^{-1}(U \cap W)\right)$. In other words, a local function on X is regular, if it induces a local regular functions on each affine chart. Regular functions $U \longrightarrow \Bbbk$ form a commutative ring $\mathscr{O}_X(U)$; a correspondence $U \longmapsto \mathscr{O}_X(U)$ is called ***a structure sheaf*** on X. More generally, a map of algebraic manifolds $X \xrightarrow{\varphi} Y$ is called ***a regular morphism***, if its pull back is well defined homomorphism of $\Bbbk$-algebras: $\mathscr{O}_Y(U) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}U)$ for any open $U \subset Y$. For example, a set of regular morphisms $X \longrightarrow \mathbb{A}^1$ coincides with $\mathscr{O}_X(X)$.

12.5. Rational maps. A regular[1] morphism $U \xrightarrow{\varphi} Y$, which is defined only on some open dense $U \subset X$, is called ***a rational map*** from X to Y. One should be careful in composing rational maps, because an image of the first map may be completely outside the domain where the second is defined.

12.5.1. Example: a projection $\mathbb{A}^{n+1} \xrightarrow{\pi} \mathbb{P}_n$ sending a point $A \in \mathbb{A}^n$ to the line $(OA) \in \mathbb{P}_n$ is a rational surjection defined on $U = \mathbb{A}^n \setminus O$. In terms of the standard affine chart

$$\mathbb{A}^n \xrightarrow{\varphi_i} U_i = \{(t_0, t_1, \dots, t_n) \in \mathbb{P}_n \mid t_i = 1\},$$

the pull back homomorphism $\mathscr{O}_{\mathbb{P}_n}(U_i) \xrightarrow{\pi^*} \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$ sends

$$f(x_1, x_2, \dots, x_n) \in k[x_1, x_2, \dots, x_n] = \mathscr{O}_{\mathbb{P}_n}(U_i)$$

to the rational function

$$\widetilde{f}(t_0, t_1, \dots, t_n) = f(t_0/t_i, \dots, t_{i-1}/t_i, t_{i+1}/t_i, \dots, t_n/t_i) \in \mathscr{O}_{\mathbb{A}^{n+1}}(\mathscr{D}(t_i)) = \mathscr{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i)).$$

12.6. Closed submanifolds. Any closed subset $Z \subset X$ of an algebraic manifold X has natural structure of algebraic manifold. Namely, for any affine chart U the intersection $Z \cap U$ is a closed subset of U, that is an affine algebraic set $\operatorname{Spec_m}(\mathscr{O}_X(U)/\mathscr{I}_Z(U))$, where $\mathscr{I}_Z(U) = \{f \in \mathscr{O}_X(U) \mid f|_Z \equiv 0\}$ is the ideal of $Z \cap U$ on U. The correspondence $U \longmapsto \mathscr{I}_Z(U)$ is called ***the ideal sheaf*** of the closed submanifold $Z \subset X$. This is a subsheaf of the structure sheaf. It consists of all local regular functions vanishing along Z. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called ***a closed embedding***, if $\varphi(X) \subset Y$ is a closed submanifold and φ gives an isomorphism between X an $\varphi(X)$. One can say that an algebraic manifold X is affine iff it admits a closed embedding into affine space. Similarly, a manifold X is called ***projective***, if it admits a closed embedding $X \hookrightarrow \mathbb{P}_m$ for some m.

12.6.1. Example: closed submanifold $X \subset Y$ is separable as soon Y is, because the diagonal in $X \times X$ is a preimage of the diagonal in $Y \times Y$ under an embedding $X \times X \hookrightarrow Y \times Y$. In particular, any affine or projective manifold is separable and has a finite type.

12.6.2. Example: graph of morphism. Let $X \xrightarrow{\varphi} Y$ be a regular morphism. A preimage of the diagonal $\Delta \subset Y \times Y$ under an induced morphism $X \times Y \xrightarrow{\varphi \times \mathrm{Id}_Y} Y \times Y$ is called ***a graph*** of φ and is denoted by Γ_φ. Geometrically, $\Gamma_\varphi = \{(x, f(x)\} \subset X \times Y$. It is closed, if Y is separable. For example, a graph of a regular morphism of affine manifolds $\operatorname{Spec_m}(A) \xrightarrow{\varphi} \operatorname{Spec_m}(B)$ is given in $A \otimes B$ by the equation system $1 \otimes f = \varphi^*(f) \otimes 1$, where f runs through B and $B \xrightarrow{\varphi^*} A$ is the pull back of φ.

12.6.3. Example: family of closed submanifolds. Any regular morphism $X \xrightarrow{\pi} Y$ may be considered as a family of closed submanifolds $X_y = \pi^{-1}(y) \subset X$ parameterized by the points $y \in Y$. If $X \xrightarrow{\pi} Y$, $X' \xrightarrow{\pi'} Y$

[1] as above, the regularity means that the pull back $\mathscr{O}_Y(W) \xrightarrow{\varphi^*} \mathscr{O}_X(\varphi^{-1}(W))$ is a well defined $\Bbbk$-algebra homomorphism

are two families with the same base, then a regular morphism $X \xrightarrow{\varphi} X'$ is called *a morphism of families* (or a morphism *over* Y), if it sends X_y to X'_y for any $y \in Y$, i.e. if $\pi = \pi' \circ \varphi$. A family $X \xrightarrow{\pi} Y$ is called *constant* or *trivial* with a fiber F, if it is isomorphic over Y to the direct product $F \times Y \xrightarrow{\pi_Y} Y$.

12.6.4. Example: blow up a point $p \in \mathbb{P}_n$. Lines passing through a given point $p \in \mathbb{P}_n$ form a projective space $E \simeq \mathbb{P}_{n-1}$, which can be identified with any hyperplane $H \subset \mathbb{P}_n$ such that $p \notin H$. An incidence graph $\mathscr{B}_p = \{(\ell, q) \in E \times \mathbb{P}_n \mid q \in \ell\}$ is called *a blow up* of $p \in \mathbb{P}_n$.

If $n \geqslant 2$, then the projection $\sigma_p : \mathscr{B}_p \twoheadrightarrow \mathbb{P}_n$ is bijective outside $q = p$, but a preimage $\sigma_p^{-1}(p) \subset \mathscr{B}_p$ coincides with E; this fiber is called *an exceptional divisor*.

The second projection $\varrho_E : \mathscr{B}_p \twoheadrightarrow E$ fibers $\mathscr{B}_p$ as *a line bundle* over E. This line bundle is called *a tautological line bundle* over E. Its fiber ϱ_E^ℓ over a point $\ell \in E$ coincides with the line $\ell \subset \mathbb{P}_n$ itself.

If we fix homogeneous coordinates $(t_0 : t_1 : \ \dots \ : t_n)$ on $\mathbb{P}_n$ such that $p = (1 : 0 : \cdots : 0)$ and identify E with the hyperplane $H = \{(0 : q_1 : \cdots : q_n)\} \subset \mathbb{P}_n$, then $(q, t) \in \mathscr{B}_p$ iff $q_i t_j = q_j t_i$ for all $1 \leqslant i < j \leqslant n$, i.e. iff

$$\operatorname{rk} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & q_1 & \cdots & q_n \\ t_0 & t_1 & \cdots & t_n \end{pmatrix} = 2 \,.$$

Thus, $\mathscr{B}_p$ is closed submanifold of $H \times \mathbb{P}_n$.

12.7. Closed morphisms.

A regular morphism $X \xrightarrow{\varphi} Y$ is called *closed*, if $\varphi(Z) \subset Y$ is closed for any closed $Z \subset X$. Of course, any closed embedding is closed. The theorem from n° 11.9.1 says that any finite morphism of affine manifolds is closed. By n° 8.8.1, the projection $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is also closed.

12.7.1. PROPOSITION. *If X is a projective manifold, then the projection $X \times Y \longrightarrow Y$ is closed for any manifold Y.*

Proof. Taking an affine chart on Y, we can suppose that Y is affine, i.e. that $X \times Y$ is a closed submanifold of $\mathbb{P}_m \times \mathbb{A}^n$. Then our projection is the restriction of the closed map $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ onto the closed subset $X \times Y \subset \mathbb{P}_m \times \mathbb{A}^n$. □

12.7.2. PROPOSITION. *If X is projective and Y is separable, then any morphism $X \xrightarrow{\varphi} Y$ is closed.*

Proof. Since Y is separable, the graph $\Gamma_\varphi \subset X \times Y$ is closed. $Z \times Y$ is also closed in $X \times Y$ for any closed $Z \subset X$. But $\varphi(Z)$ is the image of $\Gamma_\varphi \cap (Z \times Y)$ under the projection $X \times Y \longrightarrow Y$. □

12.7.3. COROLLARY. *If X is a connected projective manifold, then $\mathscr{O}_X(X) = \Bbbk$. Moreover, each regular map from X to any affine manifold contracts X into one point.*

Proof. Let us identify $k = \mathbb{A}^1$ with an affine chart on $\mathbb{P}_1$ and consider a global regular function $X \xrightarrow{f} \Bbbk$ as a regular morphism $X \xrightarrow{f} \mathbb{P}_1$. Since $f(X) \subsetneq \mathbb{P}_1$ is closed and connected, it is one point. In particular, if $X \xrightarrow{\varphi} \mathbb{A}^n$ is regular, then each coordinate form $x_i \in \Bbbk[\mathbb{A}^n]$ takes a constant value along $\varphi(X)$. □

12.8. Finite morphisms of manifolds.

A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *finite*, if $W = \varphi^{-1}(U)$ is an affine chart on X for any affine chart $U \subset Y$ and the restriction $W \xrightarrow{\varphi_W} U$ is a finite morphism of affine algebraic varieties. It follows from n° 11.9.1 that any finite morphism is closed and a restriction of a finite morphism onto any closed submanifold $Z \subset X$ is a finite morphism as well. Moreover, if X is irreducible, then each proper closed $Z \subset X$ goes to a *proper* closed subset of Y.

12.8.1. Example: a projection of any projective manifold $X \subsetneq \mathbb{P}_n$ from any point $p \notin X$ onto any hyperplane $H \not\ni p$ is a finite morphism. To check this, let us fix the coordinates as in n° 12.6.4 and follow the notations of that example. Consider a standard affine chart on H, say $U_{q_n} = \{q = (0 : u_1 : \cdots : u_{n-1} : 1)\} \subset H$. Its preimage $Y = \pi_p^{-1}(U_{q_n}) \subset X$ under the projection from p lies inside the cone over U_{q_n} with the punctured vertex p (because $p \notin X$). The blow up maping σ_p identifies this punctured cone with the affine space $\mathbb{A}^n = U_{q_n} \times \mathbb{A}^1$ via the substitution $t = \vartheta p + q_u$, where $t = (t_0 : t_1 : \ \dots \ : t_n)$ is the homogeneous coordinate on $\mathbb{P}_n$, $p = (1 : 0 : \ \dots \ : 0)$, $q_u = (0 : u_1 : \cdots : u_{n-1} : 1) \in U_{q_n}$. If X is given by a system of homogeneous equations $f_\nu(t) = 0$, then Y is given in affine coordinates (u, t) by equations

$$f_\nu(\vartheta p + q_u) = \alpha_0^{(\nu)}(u)\,\vartheta^m + \alpha_1^{(\nu)}(u)\,\vartheta^{m-1} + \cdots + \alpha_m^{(\nu)}(u) = 0 \,. \tag{12-1}$$

So, Y is affine and it remains to check that $\Bbbk[Y] = \Bbbk[u][\vartheta]/(f_\nu(\vartheta p + q_u))$ is a finitely generated $\Bbbk[u]$-module. To this aim it is enough to find an integer equation with $\alpha_0(u) \equiv 1$ in the ideal spanned by equations (12-1). Then already a factorization through this equation leads to a finitely generated $\Bbbk[u]$-module.

Note that the leading coefficients $\alpha_0^{(\nu)}(u)$ have no common zeros in U_{q_n}. Indeed, if all $\alpha_0^{(\nu)}(u)$ vanish at $u = u_0$, then the homogeneous versions of (12-1)

$$\alpha_0^{(\nu)}(u_0)\,\vartheta_0^m + \alpha_1^{(\nu)}(u_0)\,\vartheta_0^{m-1}\vartheta_1 + \cdots + \alpha_m^{(\nu)}(u_0)\vartheta_1^m = 0$$

(they are obtained by substitution $t = \vartheta_0\, p + \vartheta_1\, q_u$ and describe the intersection $X \cap (pq_{u_0})$) have the solution $(\vartheta_0 : \vartheta_1) = (1 : 0)$, which corresponds to the point $p \notin X$.

Thus, the ideal spanned by the leading coefficients $\alpha_0^{(\nu)}(u)$ is non proper and contains the unity as required.

Exercise 12.3. Check that a composition of finite morphisms is finite and prove that any projective manifold admits a finite surjective morphism onto a projective space.

12.8.2. COROLLARY. *Each affine manifold X admits a finite surjective morphism φ onto appropriate affine space $\mathbb{A}^m$.*

Proof. Let $X \subsetneq \mathbb{A}^n$, where $\mathbb{A}^n$ is embedded in $\mathbb{P}_n$ as the standard chart U_0. We write H_∞ for $\mathbb{P}_n \setminus U_0$ and $\overline{X} \subset \mathbb{P}_n$ for the projective closure of X. A projection of $\overline{X}$ from any point $p \in H_\infty \setminus \big(\overline{X} \cap H_\infty\big)$ onto any hyperplane $L \not\ni p$ induces a finite morphism from $X = \overline{X} \setminus \big(\overline{X} \cap H_\infty\big)$ to $\mathbb{A}^{n-1} = L \setminus (L \cap H_\infty)$. If it is non surjective, we repeat this procedure. □

12.9. Dimension. For an arbitrary algebraic manifold X and an arbitrary point $x \in X$, the maximal $n \in \mathbb{N}$ such that there exists a chain of irreducible closed subsets

$$\{x\} = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{n-1} \subsetneq X_n \subset X\,. \tag{12-2}$$

is called *a dimension* of X at x and is denoted by $\dim_x X$.

Certainly, if X is irreducible, then $X_n = X$ in the maximal chain (12-2). If X is reducible, then $\dim_x X$ coincides with the maximal dimension of irreducible components passing through x.

Exercise 12.4. Show that $\dim_p X = \dim_p U$ for any affine chart $U \ni p$.

HINT. Let $X_1, X_2 \subset X$ be two closed irreducible subsets and $U \subset X$ be an open set such that both $X_1 \cap U$, $X_2 \cap U$ are nonempty. Then $X_1 = X_2 \iff X_1 \cap U = X_1 \cap U$, because $X_i = \overline{X_i \cap U}$.

12.9.1. LEMMA. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of irreducible manifolds. Then $\dim_x X \leqslant \dim_{\varphi(x)} Y$ for any $x \in X$ and the equality holds iff $\varphi(X) = Y$.*

Proof. By ex. 12.4, we can assume that both X, Y are affine. Then n° 11.9.1 implies that each chain (12-2) in X produces a chain $\cdots \subsetneq \varphi(X_i) \subsetneq \varphi(X_{i+1}) \subsetneq \cdots$ of closed irreducible submanifolds in Y. Vice versa, if $\varphi(X) = Y$, then given a chain $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_{n-1} \subsetneq Y_n = Y$, for each i we can choose an irreducible component X_i of $\varphi^{-1}(Y_i)$ mapped surjectively onto Y_i. This gives a chain (12-2) in X. □

12.9.2. PROPOSITION. $\dim_p \mathbb{A}^n = n$ *at any* $p \in \mathbb{A}^n$.

Proof. Clearly, $\dim \mathbb{A}^0 = 0$. Suppose inductively that $\dim \mathbb{A}^{n-1} = n - 1$. Since any proper closed $X \subset \mathbb{A}^n$ has a finite projection on $\mathbb{A}^{n-1}$, the above lemma implies that $\dim_p X \leqslant (n-1)$ for any p. Thus, $\dim_p \mathbb{A}^n \leqslant n$. On the other hand, there is a chain (12-2) consisting of affine subspaces passing through p. So, $\dim_p \mathbb{A}^n \geqslant n$. □

12.9.3. COROLLARY. *Let X be an irreducible affine manifold and $X \xrightarrow{\varphi} \mathbb{A}^m$ be a surjective finite morphism. Then $\dim_p X = m$ at each $p \in X$. In particular, m doesn't depend on a choice of φ and $\dim_p X$ is the same for all $p \in X$.* □

Exercise 12.5. Prove that $\dim(X \times Y) = \dim X + \dim Y$ for any irreducible manifolds X, Y.

Exercise 12.6. Let $V(f) \subset \mathbb{A}^n$ be given by irreducible $f \in k[x_1, x_2, \dots, x_n]$. Show that $\dim V(f) = n - 1$.

HINT. Find a surjective finite projection $V(f) \longrightarrow \mathbb{A}^{n-1}$ (comp. with ex. 12.3 and n° 12.8.2).

12.9.4. LEMMA. *If X is irreducible, then $\dim_p V(f) = \dim_p(X) - 1$ for any non constant $f \in \mathscr{O}_X(X)$ and any $p \in V(f)$.*

Proof. We can assume that X is affine. Fix some finite surjection $X \xrightarrow{\pi} \mathbb{A}^m$ and consider the induced map

$$\psi = \pi \times f \,:\, X \xrightarrow{x \mapsto (\pi(x), f(x))} \mathbb{A}^m \times \mathbb{A}^1$$

as in the proof from n° 11.10.1. It maps X finitely and surjectively onto affine hypersurface $V(\mu_f) \subset \mathbb{A}^m \times \mathbb{A}^1$, where

$$\mu_f(u,t) = t^n + \alpha_1(u)\, t^{n-1} + \cdots + \alpha_n(u) \in \Bbbk[u_1, u_2, \dots, u_m][t]$$

is the minimal polynomial for f over $\Bbbk(\mathbb{A}^m)$. Write $H \subset \mathbb{A}^m \times \mathbb{A}^1$ for a hyperplane given by equation $t = 0$. Then $V(f) = \psi^{-1}\,(H \cap V(\mu_f))$. The intersection $H \cap V(\mu_f)$ is a hypersurface of H given in $H = \mathbb{A}^n$ by equation $\alpha_n(u) = 0$. Thus, there is a finite surjection $V(f) \xrightarrow{\psi} V(a_n) \subset \mathbb{A}^n$. Now the proposition follows from ex. 12.6 and n° 12.9.1. □

12.9.5. COROLLARY. $\dim_p V(f) \geqslant \dim_p(X) - 1$ *for any algebraic manifold* X*, an arbitrary* $f \in \Bbbk[X]$*, and any* $p \in V(f)$. □

12.9.6. COROLLARY. *For any two closed submanifolds* $X_1, X_2 \subset \mathbb{A}^n$ *and any* $x \in X_1 \cap X_2$

$$\dim_x(X_1 \cap X_2) \geqslant \dim_x(X_1) + \dim_x(X_2) - n\ .$$

Proof. Write $X_1 \xhookrightarrow{\varphi_1} \mathbb{A}^n$, $X_2 \xhookrightarrow{\varphi_2} \mathbb{A}^n$ for the corresponding closed embeddings. Then $X_1 \cap X_2$ is the preimage of diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ under the map $X_1 \times X_2 \xhookrightarrow{\varphi_1 \times \varphi_2} \mathbb{A}^n \times \mathbb{A}^n$. It is given inside $X_1 \times X_2$ by the pull backs of n equations $x_i = y_i$, which define Δ inside $\mathbb{A}^n \times \mathbb{A}^n$. It remains to apply n° 12.9.5. □

12.9.7. COROLLARY. *If* $\dim(X_1) + \dim(X_2) \geqslant n$ *for some closed* $X_1, X_2 \subset \mathbb{P}_n$*, then* $X_1 \cap X_2 \neq \varnothing$.

Proof. Let $\mathbb{P}_n = \mathbb{P}(V)$. Consider affine cones[1] $X_1', X_2'' \subset \mathbb{A}^{n+1} = \mathbb{A}(V)$ formed by the lines passing through the origin $O \in \mathbb{A}^{n+1}$ and belonging to X_1, X_2 respectively. By the previous corollary, $\dim_O(X_1' \cap X_2'') \geqslant \dim_O(X_1) + 1 + \dim_O(X_2) + 1 - n - 1 \geqslant 1$. So, $X_1' \cap X_2''$ is exhausted by O. □

12.9.8. THEOREM. *Let* $X \xrightarrow{\varphi} Y$ *be a dominant morphism of irreducible manifolds. Then:*

(1) $\dim_x \varphi^{-1}(\varphi(x)) \geqslant \dim X - \dim Y$ *for any* $x \in X$*;*

(1) *there exists open dense* $U \subset Y$ *such that* $\dim \varphi^{-1}(y) = \dim X - \dim Y$ *for all* $y \in U$.

Proof. In (1) we can replace Y by an affine neighborhood of $\varphi(x)$, i. e. assume that Y is affine. Appropriate finite projection $Y \twoheadrightarrow \mathbb{A}^m$ reduces (1) to the case $Y = \mathbb{A}^m = \operatorname{Spec_m} k[u_1, u_2, \dots, u_m]$, $\varphi(x) = 0$. Now $\varphi^{-1}(0)$ is an intersection of m hypersurfaces $V(\varphi^*(u_i))$ in X and the required inequality follows inductively from n° 12.9.5.

In (2) we can suppose that both X, Y are affine and φ is obtained by restricting the projection $Y \times \mathbb{A}^m \xrightarrow{\pi} Y$ onto some closed submanifold $X \subset Y \times \mathbb{A}^m$ (comp. with the decomposition (11-1) from ex. 11.6). Now we are going to apply the arguments from n° 12.8.2 fiberwise over Y.

Namely, consider the closure $\overline{X} \subset Y \times \mathbb{P}_m$ and choose a hyperplane $H \subset \mathbb{P}_m$ and a point $p \in \mathbb{P}_m \setminus H$ such that the section $Y \times \{p\} \subset Y \times \mathbb{P}_m$ is not contained in $\overline{X}$. The fiberwise projection from p onto H is well defined over an open subset $U \subset Y$ complementary to $\overline{\pi}\Big((Y \times \{p\}) \cap \overline{X}\Big)$, where $\overline{\pi} : Y \times \mathbb{P}_m \twoheadrightarrow Y$ is the projection along $\mathbb{P}_m$ (this is a closed morphism). Replacing Y by any non empty principal open subset of U, we get a finite morphism $X \longrightarrow Y \times \mathbb{A}^{m-1}$. After a number of such replacements we can suppose that φ is a finite surjection $X \twoheadrightarrow Y \times \mathbb{A}^n$ followed by the projection $Y \times \mathbb{A}^n \twoheadrightarrow Y$. Now all fibers have dimension $n = \dim X - \dim Y$. □

12.9.9. COROLLARY (CHEVALLEY'S SEMI-CONTINUITY THEOREM). *For any morphism of algebraic manifolds* $X \xrightarrow{\varphi} Y$ *and each* $k \in \mathbb{Z}$ *a subset* $X_k \overset{\text{def}}{=} \{\, x \in X \mid \dim_x \varphi^{-1}(\varphi(x)) \geqslant k \,\}$ *is closed in* X
.

Proof. We can suppose that X, Y are irreducible. If $k \leqslant \dim(X) - \dim(Y)$, then $X_k = X$ by the above theorem. For $k > \dim(X) - \dim(Y)$ we can replace Y by $Y' = Y \setminus U$, where U is the same as in n° 12.9.8, and X — by $X' = \varphi^{-1}(Y')$. Clearly, $X_k \subset X'$ and we can repeat the arguments decreasing the dimensions of X, Y. □

Exercise 12.7. Show that isolated points in the fibers of a morphism $X \xrightarrow{\varphi} Y$ fill an open subset in X.

12.9.10. COROLLARY. *For any closed morphism* $X \xrightarrow{\varphi} Y$ *and each* $k \in \mathbb{Z}$ *a subset*

$$Y_k \overset{\text{def}}{=} \{\, y \in Y \mid \dim \varphi^{-1}(y) \geqslant k \,\}$$

[1] they have the same equations as X_1, X_2 but these equations are considered now as affine rather than homogeneous

is closed in Y. □

12.9.11. COROLLARY. *Let $X \xrightarrow{\varphi} Y$ be a closed morphism with irreducible fibers of the same dimension. Then irreducibility of Y implies that X is irreducible as well.*

Proof. Let $X = X' \cup X''$. Since each fiber $\varphi^{-1}(y)$ is irreducible, it completely belongs to one of X', X''. Applying n° 12.9.10 to the restriction $X' \xrightarrow{\varphi|_{X'}} Y$, we see that a set of all fibers completely contained in X' is mapped to some closed subset $Z' \subset Y$. Similarly, all fibers completely contained in X'' are also mapped to some closed $Z'' \subset Y$. So, $Y = Z' \cup Z''$ but both Z', Z'' should be proper as soon as X', X'' were proper. □

§13. Working example: lines on surfaces.

13.1. Variety of lines on surfaces of given degree. We are going to analyze the set of lines lying on a surface $S \subset \mathbb{P}_3$ of a given degree d.

Exercise 13.1. Carry out the complete analysis for $d = 2$.

To this aim consider the space $\mathbb{P}_N = \mathbb{P}(S^d V^*)$, of surfaces of degree d in $\mathbb{P}_3 = \mathbb{P}(V)$, and identify the set of all lines in $\mathbb{P}(V)$ with the Plücker quadric $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Let

$$\Gamma \stackrel{\text{def}}{=} \{ (S, \ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \} \subset \mathbb{P}_N \times Q_P$$

be the incidence graph.

13.1.1. CLAIM. *Γ is closed submanifold of $\mathbb{P}_N \times Q_P$.*

Proof. 2-dimensional subspace spanned by $u, w \in V$ coincides with the image of the contraction map $V^* \longrightarrow V$, which sends $\xi \in V^*$ to $\langle \xi , u \wedge w \rangle$. So, the line $\ell = (uw)$ lies on a surface S given by $F = 0$ iff $F(\langle \xi , u \wedge w \rangle) \cong 0$ identically in $\xi \in V^*$. In coordinates, let e_ν form a basis of V, ξ_ν be the coordinates of ξ w.r.t. the dual basis of V^*, and $u \wedge w = \sum\limits_{\mu \neq \nu} p_{\mu\nu} e_\mu \wedge e_\nu$, where $p_{\mu\nu} = -p_{\nu\mu}$ are the associated Plücker coordinates on $\mathbb{P}_5 \supset Q$. Then $\langle \xi , u \wedge w \rangle = \sum\limits_i \xi_i \cdot (\sum_\nu p_{i\nu} e_\nu)$. Substitute this into F, expand the result through the monomials in ξ, and write down that all coefficients of this expansion vanish — this gives a system of polynomial equations on the coefficients of F and p_{ij} describing $\Gamma \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$. □

13.1.2. CLAIM. *Projection $\Gamma \xrightarrow{\pi_2} Q_P$ is surjective; all its fibers are projective spaces of dimension $d(d+1)(d+5)/6 - 1$.*

Proof. Let a line $\ell \subset \mathbb{P}(V)$ be given by $x_0 = x_1 = 0$. Then $S \supset \ell$ iff S has an equation $0 = x_2 \cdot F_2(x) + x_3 \cdot F_3(x)$, where $F_2, F_3 \in S^{d-1}V^*$ are arbitrary homogeneous polynomials. These equations form a vector space W, which coincides with the image of the linear operator $S^{d-1}V^* \oplus S^{d-1}V^* \xrightarrow{(f,g) \mapsto x_2 f + x_3 g} S^d V^*$ whose kernel consists of all (f, g) such that $x_2 f = -x_3 g$ that is possible iff $f = x_3 h$ and $g = -x_2 h$ for some $h \in S^{d-2}V^*$. Hence, the kernel is isomorphic to $S^{d-2}V^*$ and $\dim W = 2 \dim(S^{d-1}V^*) - \dim(S^{d-1}V^*) = \frac{1}{6}\Big(2\, d(d+1)(d+2) - (d-1)d(d+1)\Big) = d(d+1)(d+5)/6$. □

13.1.3. COROLLARY. *Γ is an irreducible projective manifold of dimension $d(d+1)(d+5)/6 + 3$.*

Proof. This follows at once from n° 12.9.11 and n° 12.9.8. □

13.1.4. CLAIM. *A generic[1] surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines.*

Proof. By n° 12.7.2, the image of the projection $\Gamma \xrightarrow{\pi_1} \mathbb{P}_N$, that is the set of all surfaces containing some lines, is closed irreducible submanifold of $\mathbb{P}_N = \mathbb{P}(S^d V^*)$. By n° 12.9.8, its dimension equals $\dim \Gamma$ minus the minimal dimension of non-empty fibres of π_1. We see that the image is proper as soon $\dim \Gamma < N$, i.e. when

$$d(d+1)(d+5)/6 + 3 < (d+1)(d+2)(d+3)/6 \,.$$

This holds for all $d \geqslant 4$. □

13.1.5. CLAIM. *Each cubic surface $S_3 \subset \mathbb{P}_3$ contains lines; generically, this is a finite set of lines.*

Proof. Taking in the previous proof $d = 3$, we get $\dim \Gamma = N = 19$. Thus, to show that π_1 is surjective, it is enough to find a non-empty 0-dimensional fiber of π_1, i.e. to present a cubic surface containing a finite set of lines.

Let us find all the lines, say, on a cubic C with affine equation $xyz = 1$. This affine piece does not contain the lines at all, because $x = x_0 + \alpha t$, $y = y_0 + \beta t$, $z = z_0 + \gamma t$ lies on C iff $\alpha\beta\gamma = 0$, $\alpha\beta z_0 + \beta\gamma x_0 + \gamma\alpha y_0 = 0$, and $\alpha y_0 z_0 + \beta x_0 z_0 + \gamma x_0 y_0 = 0$, but $x_0 y_0 z_0 = 1$, which leads to contradiction when we go from the left to the right: for example, $\alpha = 0 \Rightarrow \beta = 0$ or $\gamma = 0 \Rightarrow \beta = \gamma = 0$. To describe C at infinity, put $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$ and rewrite its equation as $x_1 x_2 x_3 = x_0^3$. Thus, $C \cap \{x_0 = 0\}$ consists of 3 lines: $x_i = x_0 = 0$, $i = 1, 2, 3$. □

Exercise 13.2*. Find all lines on the (smooth) Fermat cubic C_F, given by $\sum x_i^3 = 0$.

[1] at least any one from some dense open subset in the space of all degree d surfaces

HINT. C_F is preserved by the permutations of the coordinates; up to permutations, a pair of linear equations for $\ell \subset C_F$ can be reduced by the Gauss method to $x_0 = \alpha x_2 + \beta x_3$, $x_1 = \gamma x_2 + \delta x_3$; substitute this in Fermat's cubic equation, show that $\alpha\beta\gamma\delta = 0$ e. t. c.

13.2. Lines on a smooth cubic. Now, let $S \subset \mathbb{P}_3$ be a smooth cubic surface with equation $F(x) = 0$.

13.2.1. LEMMA. *A reducible plane section of S can split either into a line and a smooth conic or into a triple of distinct lines.*

Proof. We have to show that a plane section $\pi \cap S$ can not contain a double line component. If there is a double line $\ell \subset \pi \cap S$, we can take the coordinates where π is given by $x_2 = 0$ and ℓ is given by $x_2 = x_3 = 0$. Then $F(x) = x_2 Q(x) + x_3^2 L(x) = 0$ for some linear L and quadratic Q. Let a be an intersection point of ℓ with the quadric $Q(x) = 0$. Then $x_2(a) = x_3(a) = Q(a) = 0$ implies that all partial derivatives $\partial F/\partial x_i$ vanish at a, i.e. S is singular at a. □

13.2.2. COROLLARY. *A point of S can belong to at most 3 lines lying on S and these lines should be coplanar.*

Proof. Indeed, all lines passing through $p \in S$ and lying on S belong to $S \cap T_pS$. □

13.2.3. LEMMA. *Given $\ell \subset S$, there are precisely 5 distinct planes $\pi_1, \pi_2, \dots, \pi_5$ containing ℓ and intersecting S in a triple of lines; moreover, if $\pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$, then $\ell_i \cap \ell_j = \ell_i \cap \ell_j' = \ell_i' \cap \ell_j' = \varnothing$ $\forall\, i \neq j$ (in particular, S contains some skew lines) and any line on S skew to ℓ must intersect for each i precisely one of ℓ_i, ℓ_i'.*

Proof. Fix a basis $\{e_0, e_1, e_2, e_3\}$ for V such that $\ell = (e_0 e_1)$, given by equations $x_2 = x_3 = 0$, lies on S. Then the equation $F(x) = 0$, defining S, can be written in this basis as:

$$L_{00}(x_2,x_3)\cdot x_0^2 + 2\,L_{01}(x_2,x_3)\cdot x_0x_1 + L_{11}(x_2,x_3)\cdot x_1^2 + 2\,Q_0(x_2,x_3)\cdot x_0 + 2\,Q_1(x_2,x_3)\cdot x_1 + R(x_2,x_3) = 0 \quad (13\text{-}1)$$

where $L_{ij}, Q_\nu, R \in k[x_2,x_3]$ are homogeneous of degrees $1,2,3$ respectively. Let us parameterize a pencil of plains passing through ℓ by the points $e_\vartheta = \vartheta_2 e_2 + \vartheta_3 e_3 \in (e_2 e_3)$ and write $(t_0 : t_1 : t_2)$ for homogeneous coordinates in the plane $\pi_\vartheta = (e_0 e_1 e_\vartheta)$ w.r.t. these basic points. An equation for the plane conic $(\pi_\vartheta \cap S) \setminus \ell$ is obtained from (13-1) by the substitution $x = (t_0 : t_1 : \vartheta_2 t_3 : \vartheta_3 t_3)$ and canceling the common factor t_3. The resulting conic has the Gram matrix

$$G = \begin{pmatrix} L_{00}(\vartheta) & L_{01}(\vartheta) & Q_0(\vartheta) \\ L_{01}(\vartheta) & L_{11}(\vartheta) & Q_1(\vartheta) \\ Q_0(\vartheta) & Q_1(\vartheta) & R(\vartheta) \end{pmatrix}$$

whose determinant is homogeneous degree 5 polynomial in $\vartheta = (\vartheta_2 : \vartheta_3)$

$$D(\vartheta_2,\vartheta_3) = L_{00}(\vartheta)L_{11}(\vartheta)R(\vartheta) + 2\,L_{01}(\vartheta)Q_0(\vartheta)Q_1(\vartheta) - L_{11}(\vartheta)Q_0^2(\vartheta) - L_{00}(\vartheta)Q_1^2(\vartheta) - L_{01}(\vartheta)^2R(\vartheta)\,.$$

Thus, it has 5 roots counted with multiplicities. We have to show that all these roots are simple. Each root corresponds to a splitting of the conic into a pair of lines ℓ', ℓ''. There are two possibilities: the intersection point $\ell' \cap \ell''$ lies either on ℓ or outside ℓ.

In the first case, we can fix a basis in order to have $\ell' = (e_0 e_2)$ and $\ell'' = \big(e_0\,(e_1 + e_2)\big)$. These lines are given by equations $x_3 = x_1 = 0$ and $x_3 = (x_1 - x_2) = 0$. Such the splitting corresponds to the root $\vartheta = (1:0)$. Its multiplicity equals the highest power of ϑ_3 dividing $D(\vartheta_2,\vartheta_3)$. Since $\ell, \ell', \ell'' \subset S$, the equation (13-1) has a form $x_1x_2(x_1 - x_2) + x_3 \cdot q(x)$ with some quadratic $q(x)$. Thus, elements of G that may be not divisible by ϑ_3 are exhausted by $L_{11} \equiv x_2 \pmod{\vartheta_3}$ and $Q_1 \equiv -x_2^2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2,\vartheta_3) \equiv -L_{00}Q_1^2 \pmod{\vartheta_3^2}$. This term is of order 1 in t_3 as soon $x_1x_2^2$ and $x_0^2x_2$ do come in (13-1) with non zero coefficients. But the first is the only monomial that gives non zero contribution into $\partial F/\partial x_1$ computed at $e_2 \in S$ and the second — in $\partial F/\partial x_2$ at $e_0 \in S$. Hence, they do come.

In the second case we fix a basis in order to have $\ell' = (e_0 e_2)$, $\ell'' = (e_1 e_2)$, which are given by equations $x_3 = x_1 = 0$ and $x_3 = x_0 = 0$. This splitting corresponds to the same root $\vartheta = (1:0)$. Now equation (13-1) turns to $x_0x_1x_2 + x_3 \cdot q(x)$ and non zero modulo ϑ_3 entry of G is only $L_{01} \equiv x_2/2 \pmod{\vartheta_3}$. Thus, $D(\vartheta_2,\vartheta_3) \equiv -L_{01}^2 R$ $\pmod{\vartheta_3^2}$, which is of the first order in t_3 as soon $x_2^2x_3$ and $x_0x_1x_2$ do really appear in (13-1). The second does, because otherwise F is divisible by x_3. The first is the only monomial that gives non zero contribution into $\partial F/\partial x_3$ computed at $e_2 \in S$.

The rest assertions follow immediately from n° 13.2.2, n° 13.2.1 and remark that any line in $\mathbb{P}_3$ intersects any plane. □

13.2.4. LEMMA. *Any four mutually skew lines on S do not lie simultaneously on a quadric and there exist either one or two (but no more!) lines on S intersecting each of these four lines.*

Proof. If four given lines on S lie on some quadric Q, then Q is smooth and the lines belong to the same line family[1] ruling this quadric. Each line from the second ruling family on Q lies on S, because a line passing through 4 distinct points of S has to lie on S. Hence, $Q \subset S$ and S is reducible. It remains to apply ex. 2.4. □

13.3. Configuration of 27 lines. Take 2 skew lines $a, b \subset S$ and construct 5 pairs of lines ℓ_i, ℓ'_i predicted by n° 13.2.3 applied to $\ell = a$. Let us write ℓ_i for those lines that do meet b and ℓ'_i for remaining lines, which do not. There are 5 more lines ℓ''_i coupled with ℓ_i by n° 13.2.3 applied to $\ell = b$. Each ℓ''_i meets b but neither a nor ℓ_j with $j \neq i$. Thus, ℓ''_i intersects all ℓ'_j with $j \neq i$.

Any line $c \subset S$, different from 17 just constructed, is skew to a, b but meets either ℓ_i or ℓ'_i for each i. By n° 13.2.4, all lines meeting $\geqslant 4$ of ℓ_i's are exhausted by a, b. Let c meet $\leqslant 2$ of ℓ_i's. Then, up to a permutation of indices, c meets ℓ'_1, ℓ'_2, ℓ'_3 and, say, either ℓ'_4 or ℓ_5. In the both cases we already have two distinct lines $a, \ell''_5 \neq c$ intersecting all these 4 lines. This contradicts to n° 13.2.4.

We conclude that c intersects precisely 3 of 5 lines ℓ_i.

13.3.1. LEMMA. *Remaining lines $c \subset S$ are in 1–1 correspondence with 15 triples*

$$\{i,\, j,\, k\} \subset \{1,\, 2,\, 3,\, 4,\, 5\}\,.$$

Proof. There is at most one line c intersecting a given triple of ℓ_i's — this is the second possible line besides a meeting all these ℓ_i's and the rest ℓ'_j's (all 5 are mutually skew). On the other hand, by n° 13.2.3, for each i there are precisely 10 lines on S intersecting ℓ_i: 4 of them are a, b, ℓ'_i, ℓ''_i and other 6 have to intersect exactly 2 of the rest four ℓ_j's. So, we have 1–1 correspondence between these 6 lines and $6 = \binom{4}{2}$ choices of pairs of ℓ's. □

Thus, we have proven

13.3.2. COROLLARY. *Each smooth cubic surface $S \subset \mathbb{P}_3$ contains precisely 27 lines and their incidence combinatorics is the same for all S.* □

Exercise 13.3. Let $\mathfrak{G} \subset \mathfrak{S}_{27}$ be a group of all permutations of the 27 lines preserving all the incidence relations between them; find the order of $\mathfrak{G}$. (ANSWER: $|\mathfrak{G}| = 51\,840 = 2^7 \cdot 3^4 \cdot 5$)

Exercise 13.4*. Consider the field of 4 elements $\mathbb{F}_4 \overset{\text{def}}{=} \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$, where $\mathbb{F}_2 = \mathbb{Z}/(2)$. The extension $\mathbb{F}_2 \subset \mathbb{F}_4$ has a conjugation automorphism[2] $z \longmapsto \overline{z} \overset{\text{def}}{=} z^2$, which lives $\mathbb{F}_2$ fixed and permutes two roots of the polynomial $\omega^2+\omega+1$. Show that *unitary*[3] 4×4 - matrices with entries in $\mathbb{F}_4$ modulo the scalar matrices form a (normal) subgroup of index 2 into the group $\mathfrak{G}$ from ex. 13.3.

HINT. The unitary group preserves the Fermat cubic C_F (see ex. 13.2) whose equation over $\mathbb{F}_4$ turns to the standard Hermitian form $\sum x_i\overline{x}_i$.

[1] comp. with n° 2.8.1–n° 2.8.2
[2] quite similar to the complex conjugation in the extension $\mathbb{R} \subset \mathbb{C}$
[3] i. e. satisfying $\overline{M} \cdot M^t = E$

§14. General nonsense appendix.

14.1. Categories. Let us evade an explicit formal definition of «a category»[1]. Informally, *a category* $\mathscr{C}$ consists of *objects*, which form a class[2] denoted by $\operatorname{Ob}\mathscr{C}$, and for each pair of objects $X, Y \in \operatorname{Ob}\mathscr{C}$ there is a set of *morphisms* $\operatorname{Hom}(X,Y) = \operatorname{Hom}_{\mathscr{C}}(X,Y)$. These sets are distinct for distinct pairs X, Y. It is convenient to think of the morphisms as the arrows $X \longrightarrow Y$. All these data have to satisfy the following properties:

- for any ordered triple of objects $X, Y, Z \in \operatorname{Ob}\mathscr{C}$ there is *a composition map*
$$\operatorname{Hom}(Y,Z) \times \operatorname{Hom}(X,Y) \xrightarrow{(\varphi,\psi)\mapsto\varphi\circ\psi} \operatorname{Hom}(X,Z)\ ,$$
which is *associative*: $(\chi\circ\varphi)\circ\psi = \chi\circ(\varphi\circ\psi)$;
- for any $X \in \operatorname{Ob}\mathscr{C}$ there is a unique[3] *identity morphism* $\operatorname{Id}_X \in \operatorname{Hom}(X,X)$ that satisfies
$$\varphi\circ\operatorname{Id}_X = \varphi\ , \quad \operatorname{Id}_X\circ\psi = \psi$$
for any morphisms $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} X$ and any $Y \in \operatorname{Ob}\mathscr{C}$.

Probably, the reader is familiar with some «big» categories like topological spaces and continuous maps as the morphisms, or finitely generated $\Bbbk$-algebras with unity and algebra homomorphisms preserving unity, or affine algebraic varieties with regular maps, e. t. c.

Of course, there are much simpler examples of categories. Say, each partially ordered set can be considered as a category in which $\operatorname{Hom}(X,Y)$ consist of one arrow, if $X \leqslant Y$, and is empty, if X and Y are non comparable. Further, any monoid M (i. e. a semigroup with unity) can be considered as a category with just one object X and $\operatorname{Hom}(X,X) = M$.

Two objects $X, Y \in \operatorname{Ob}\mathscr{C}$ of an arbitrary category are called *isomorphic*, if there are two arrows $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ (called *inverse isomorphisms*) such that $\varphi\circ\psi = \operatorname{Id}_Y$, $\psi\circ\varphi = \operatorname{Id}_X$.

Given a category $\mathscr{C}$, one can always construct *an opposite category* $\mathscr{C}^{\mathrm{opp}}$ with the same objects $\operatorname{Ob}\mathscr{C}^{\mathrm{opp}} = \operatorname{Ob}\mathscr{C}$ but inverted arrows $\operatorname{Hom}_{\mathscr{C}^{\mathrm{opp}}}(X,Y) \overset{\mathrm{def}}{=} \operatorname{Hom}_{\mathscr{C}}(Y,X)$. The duality $\mathscr{C} \leftrightarrow \mathscr{C}^{\mathrm{opp}}$ is called *reversing of arrows*.

We have seen that the category of finitely generated $\Bbbk$-algebras looks like an opposite category for the category of affine algebraic varieties over the same field $\Bbbk$. To make this statement more precise we need a tool «for comparing» the categories.

14.2. Functors are «homomorphisms of categories». More precisely, *a covariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a map $\operatorname{Ob}\mathscr{C} \xrightarrow{X\mapsto F(X)} \operatorname{Ob}\mathscr{D}$ together with a collection of maps
$$\operatorname{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \operatorname{Hom}_{\mathscr{D}}(F(X),F(Y))$$
defined for each pair $X, Y \in \operatorname{Ob}\mathscr{C}$ and preserving the compositions, i. e. satisfying
$$F(\varphi\circ\psi) = F(\varphi)\circ F(\psi)$$
as soon $\varphi\circ\psi$ is defined. Note that this forces $F(\operatorname{Id}_X) = \operatorname{Id}_{F(X)}$.

[1]like in the calculus, where «the sets» are usually successfully employed without proper logical background

[2]explicit logical formalization of this notion requires quite deep settling down into logical casuistry laying fahr enough from our current subject; we would like to consider «the category of all sets», whose objects do not form a set, certainly; but they can be described by means of appropriate «second order langauge», which exists, and that is all we need here

[3]uniqueness can be formally deduced from the defining relations, because two identity morphisms Id'_X, Id''_X satisfy $\operatorname{Id}'_X = \operatorname{Id}'_X\circ\operatorname{Id}''_X = \operatorname{Id}''_X$

Dually, *a contravariant functor* $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a covariant functor $\mathscr{C}^{\text{opp}} \xrightarrow{F} \mathscr{D}$. In other words, a covariant functor is an «antihomomorphism of categories», that is takes

$$\operatorname{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi\mapsto F(\varphi)} \operatorname{Hom}_{\mathscr{D}}(F(Y),F(X))$$

for each pair $X,Y\in\operatorname{Ob}\mathscr{C}$ and satisfies $F(\varphi\circ\psi) = F(\psi)\circ F(\varphi)$.

For example, the dualization, which takes each vector space V over $\Bbbk$ to its dual V^* and each linear map $V \xrightarrow{\varphi} W$ to the dual map $W^* \xrightarrow{\varphi^*} V^*$, is a contravariant functor from the category of vector spaces and linear maps to itself. The double dualization gives then an example of a covariant functor.

For any $\mathscr{C}$ we always have *the identity functor* $\mathscr{C} \xrightarrow{\operatorname{Id}_{\mathscr{C}}} \mathscr{C}$, which acts identically on the objects and the arrows.

An other trivial series of examples is given by *forgetful functors*. They act from categories of sets equipped with an extra structure[1] to the category $\mathscr{S}et$ of ordinary sets. Such a functor also acts identically on objects and arrows — it just forgets the extra structure.

Less trivial is

14.2.1. Example: Hom-functors. Each $X\in\operatorname{Ob}\mathscr{C}$ produces two functors from $\mathscr{C}$ to category of sets.

A covariant functor $h^X : \mathscr{C} \longrightarrow \mathscr{S}et$ takes an object Y to $h^X(Y) \overset{\text{def}}{=} \operatorname{Hom}(X,Y)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map $h^X(\varphi)\ :\ h^X(Y_1) = \operatorname{Hom}(X,Y_1) \xrightarrow{\psi\mapsto\varphi\circ\psi} \operatorname{Hom}(X,Y_2) = h^X(Y_2)$.

A contravariant functor $h_X : \mathscr{C} \longrightarrow \mathscr{S}et$ takes an object Y to $h_X(Y) \overset{\text{def}}{=} \operatorname{Hom}(Y,X)$ and an arrow $Y_1 \xrightarrow{\varphi} Y_2$ to the composition map $h_X(\varphi)\ :\ h_X(Y_2) = \operatorname{Hom}(Y_2,X) \xrightarrow{\psi\mapsto\psi\circ\varphi} \operatorname{Hom}(Y_1,X) = h_X(Y_1)$.

Exercise 14.1. Show that in the category $\mathscr{M}od(K)$, of modules over commutative ring K with K-linear morphisms, the functor h^X takes any exact triple of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ to an exact triple

$$0 \longrightarrow \operatorname{Hom}(X,A) \longrightarrow \operatorname{Hom}(X,B) \longrightarrow \operatorname{Hom}(X,C)$$

whose rightmost arrow is non surjective in general. Formulate and prove the similar property of the contravariant functor h_X.

14.3. Natural transformations. Given two (covariant) functors $F,G\ :\ \mathscr{C} \longrightarrow \mathscr{D}$, *a morphism of functors*[2] $F \overset{f}{\longmapsto} G$ is a collection of arrows $F(X) \xrightarrow{f_X} G(X) \in \operatorname{Hom}_{\mathscr{D}}(F(X),G(X))$ (parameterized by $X\in\operatorname{Ob}\mathscr{C}$) such that for any morphism $X \xrightarrow{\varphi} Y$ in $\mathscr{C}$ we have the following commutative square[3] of morphisms in $\mathscr{D}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "f_X"'] & F(Y) \arrow[d, "f_Y"] \\ G(Y) \arrow[r, "G(\varphi)"] & G(Y) \end{tikzcd} \tag{14-1}$$

For example, the canonical embedding $V \overset{i_V}{\hookrightarrow} V^{**}$ of a vector space into double dual[4] is a natural transformation from the identical functor on the category of vector spaces to the functor of double dualization.

Of course, the identity maps give an identity transformation from any functor to itself. Clearly, two natural transformations can be composed. Thus, we get

14.3.1. CLAIM. *For any two categories $\mathscr{C}$, $\mathscr{D}$ all covariant functors $\mathscr{C} \xrightarrow{F} \mathscr{D}$ form a category $\mathscr{F}un(\mathscr{C},\mathscr{D})$ whose morphisms are natural transformations of functors.* □

[1] it may be geometric, like a topology, a differentiable manifold structure e. t. c. , or algebraic, like a structure of group, ring, e. t. c. ; the morphisms in such a category are the set theoretical maps preserving this extra structure

[2] also called *a natural transformation* of functors

[3] a diagram of morphisms in a category is called *commutative*, if the compositions of arrows taken along different passes joining the same pair of vertexes always coincide

[4] sending a vector $v\in V$ to the corresponding evaluation functional $V^* \xrightarrow{\operatorname{ev}_v} \Bbbk$

14.4. Equivalence of categories. A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is called *an equivalence* of categories, if there is a functor $\mathscr{D} \xrightarrow{G} \mathscr{C}$ (called *quasi-inverse* to F) such that the composition GF is isomorphic to $\mathrm{Id}_{\mathscr{C}}$ in category $\mathscr{F}un(\mathscr{C},\mathscr{C})$ and the composition FG is isomorphic to $\mathrm{Id}_{\mathscr{D}}$ in category $\mathscr{F}un(\mathscr{D},\mathscr{D})$.

Note that our requirement «be isomorphic» to the identical functor is much weaker than another possible request «coincide with» the identical functor.

For example, consider the category $\Bbbk^n$, which has only one object — n-dimensional coordinate vector space over $\Bbbk$. The arrows in this category are linear maps $\Bbbk^n \longrightarrow k^n$. There is a natural functor $\Bbbk^n \xrightarrow{F} \mathscr{V}ect_n(\Bbbk)$, which embeds $\Bbbk^n$ to the category of *all* n-dimensional vector space over $\Bbbk$. This is an equivalence of categories. To construct (some) quasi-inverse to F functor $\mathscr{V}ect_n(k) \xrightarrow{G} \Bbbk^n$, we fix for each V some isomorphism $f_V : V \xrightarrow{\sim} \Bbbk^n$, and send an arrow $V \xrightarrow{\varphi} W$ from $\mathrm{Hom}_{\mathscr{V}ect_n(\Bbbk)}(V,W)$ to the arrow $f_W \circ \varphi \circ f_V^{-1}; \Bbbk^n \longrightarrow \Bbbk^n$. In other words, we fix some basis in each vector space and present each linear map by its matrix in these bases. Then GF coincides with the identity functor on $\Bbbk^n$. The opposite composition $FG : \mathscr{V}ect_n(\Bbbk) \longrightarrow \mathscr{V}ect_n(\Bbbk)$ is not the identity functor, because the image of FG contains just one object $\Bbbk^n \in \mathrm{Ob}\,\mathscr{V}ect_n(\Bbbk)$. But FG *is isomorphic* to the identity functor via the natural transformation provided by isomorphisms $V \xrightarrow{f_V} \Bbbk^n$.

This example has a straightforward generalization. Let $\mathscr{C} \xrightarrow{F} \mathscr{D}$ be a (covariant) functor. It is called *full*, if all maps

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\varphi \mapsto F(\varphi)} \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)) \tag{14-2}$$

are surjective. If all maps (14-2) are injective, F is called *faithful*.

14.4.1. CLAIM. *A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is an equivalence of categories iff it is full faithful and each $Y \in \mathrm{Ob}\,\mathscr{D}$ is isomorphic to $F(X)$ for some $X \in \mathrm{Ob}\,\mathscr{D}$ (depending on Y).*

Proof. For any $Y \in \mathrm{Ob}\,\mathscr{D}$ fix some isomorphism $i_Y : Y \xrightarrow{\sim} F(X)$, which exists by our assertion, and put $G(Y) = X$. For any arrow $Y_1 \xrightarrow{\varphi} Y_2$ define $G(\varphi) : G(Y_1) \longrightarrow G(Y_2)$ as an arrow that corresponds to the arrow

$$i_{Y_2} \circ \varphi \circ i_{Y_1}^{-1} \ : \ F(G(Y_1)) \longrightarrow F(G(Y_2))$$

under the isomorphisms (14-2): $\mathrm{Hom}_{\mathscr{C}}(G(Y_1),G(Y_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(FG(Y_1),FG(Y_2))$ provided by F. Remaining verifications are collected in the exercise below. □

Exercise 14.2. Check that $\mathscr{D} \xrightarrow{G} \mathscr{C}$ a) is a functor; b) is quasi-inverse to F.

14.5. Representable functors. A contravariant functor $\mathscr{C}^{\mathrm{opp}} \xrightarrow{F} \mathscr{S}et$ is called *representable*, if there exist an object $X \in \mathrm{Ob}\,\mathscr{C}$ such that in the category $\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)$ the functor

$$h_X : Y \longmapsto \mathrm{Hom}(Y,X)$$

(from n° 14.2.1) is isomorphic to F. In this case X is called *the representing object* for F. Dually, a covariant functor $\mathscr{C} \xrightarrow{F} \mathscr{S}et$ is called *corepresentable*, if in the category $\mathscr{F}un(\mathscr{C},\mathscr{S}et)$ it is isomorphic to the functor

$$h^X : Y \longmapsto \mathrm{Hom}(X,Y)$$

for some $X \in \mathrm{Ob}\,\mathscr{C}$, which is called *the corepresenting object* of F.

It is easy to see that the mapping $\varrho : A \longmapsto h_A$ gives a covariant functor $\varrho : \mathscr{C} \longrightarrow \mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)$, which sends an arrow $A \xrightarrow{\alpha} B$ in $\mathscr{C}$ to the natural transformation

$$\left(h_A \xrightarrow{\varrho(\alpha)} h_B\right) \in \mathrm{Hom}_{\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)}(\varrho(A),\varrho(B))$$

whose action over $X \in \mathrm{Ob}\,\mathscr{C}$ is $\varrho(\varphi)_X \ : \ h_A(X) = \mathrm{Hom}(X,A) \xrightarrow{\psi \mapsto \varphi \circ \psi} \mathrm{Hom}(X,B) = h_B(X)\,.$

Exercise 14.3. Check that $\varrho(\varphi)$ is a natural transformation (i. e. verify that the corresponding diagrams (14-1) are commutative), and show that $\varrho(\varphi_1 \circ \varphi_2) = \varrho(\varphi_1) \circ \varrho(\varphi_2)$.

Thus, there is a bifunctor $\mathscr{C}^{\mathrm{opp}} \times \mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et) \longrightarrow \mathscr{S}et$ that takes a pair (A,F) to the set

$$\mathrm{Hom}_{\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)}(h_A,F)\ ,$$

of all natural transformations from h_A to F. At the same time, there is the tautological evaluation bifunctor $\mathrm{ev} : \mathscr{C}^{\mathrm{opp}} \times \mathscr{F}un(\mathscr{C}^{\mathrm{opp}}, \mathscr{S}et) \longrightarrow \mathscr{S}et$, which takes (A,F) to $F(A)$. These two bifunctors are isomorphic.

14.5.1. CLAIM (YONEDA LEMMA). *For any category $\mathscr{C}$ there is an isomorphism*

$$\mathrm{Hom}_{\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)}(h_A, F) \xrightarrow{\sim} F(A) \tag{14-3}$$

functorial in $A \in \mathscr{C}$, $F \in \mathscr{F}un(\mathscr{C}^{\mathrm{opp}}, \mathscr{S}et)$. It takes a natural transformation $f : h_A \longrightarrow F$ to an element $f_A(\mathrm{Id}_A) \in F(A)$, where $\mathrm{Id}_A \in \mathrm{Hom}_{\mathscr{C}}(A,A) = h_A(A)$ is the identity and $h_A(A) \xrightarrow{f_A} F(A)$ is an action of the natural transformation f over the object A. The inverse map takes an element $a \in F(A)$ to a natural transformation

$$\left\{\mathrm{Hom}(X,A) \xrightarrow{f_X} F(X)\right\}_{X \in \mathrm{Ob}\,\mathscr{C}}$$

that sends an arrow $X \xrightarrow{\varphi} A$ to a value of the map $F(A) \xrightarrow{F(\varphi)} F(X)$ at the element a.

Proof. It is a kind of tautology. For any $X \in \mathrm{Ob}\,\mathscr{C}$ and any arrow $X \xrightarrow{\varphi} A$ we have commutative diagram (14-1)

$$\begin{tikzcd} h_A(A) = \mathrm{Hom}(A,A) \arrow[r, "h_A(\varphi)"] \arrow[d, "f_A"'] & \mathrm{Hom}(X,A) = h_A(X) \arrow[d, "f_X"] \\ F(A) \arrow[r, "F(\varphi)"] & F(X)\,. \end{tikzcd} \tag{14-4}$$

The upper map sends Id_A to φ. So, $f_X(\varphi) = F(\varphi)(f_A(\mathrm{Id}_A))$. This means that each natural transformation $h_A \xrightarrow{f} F$ is completely recovered as soon the element $a = f_A(\mathrm{Id}_A) \in F(A)$ is given, and any element $a \in F(A)$ leads to the natural transformation f defined by prescription that the diagrams (14-4) are commutative for all $X \in \mathrm{Ob}\,\mathscr{C}$. Bifunctoriality of the diagram (14-4) in A, F is evident. □

14.5.2. COROLLARY. *Functor $\mathscr{C} \xrightarrow{\varrho} \mathscr{F}un(\mathscr{C}^{\mathrm{opp}}, \mathscr{S}et) : A \longmapsto h_A$ is full and faithful*[1].

Proof. Required bifunctorial identification $\mathrm{Hom}_{\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)}(h_A, h_B) = \mathrm{Hom}_{\mathscr{C}}(A,B)$ follows from the Yoneda lemma applied to the functor $F = h_B$. □

Thus, representable functors form a full subcategory of $\mathscr{F}un(\mathscr{C}^{\mathrm{opp}}, \mathscr{S}et)$. and this subcategory is equivalent to the initial subcategory $\mathscr{C}$. In particular, a representing object (if exists) is *unique* up to *canonical* isomorphism. More precisely, given two isomorphisms

$$h_{X_1} \xleftarrow{f_1} F \xrightarrow{f_2} h_{X_2}$$

in the category $\mathscr{F}un(\mathscr{C}^{\mathrm{opp}}, \mathscr{S}et)$, then there exists a unique isomorphism $X_1 \xrightarrow{\sigma} X_2$ in the original category $\mathscr{C}$ such that for any $Y \in \mathrm{Ob}\,\mathscr{C}$ the action of natural transformation $f_2 f_1^{-1}$ over Y

$$(f_2 f_1^{-1})_Y : h_{X_1}(Y) \longrightarrow h_{X_2}(Y)$$

coincides with the composition map $\mathrm{Hom}(Y, X_1) \xrightarrow{\psi \mapsto \sigma \circ \psi} \mathrm{Hom}(Y, X_2)$.

Exercise 14.4. State and prove the dual version of the Yoneda lemma, which serves covariant functors h^A, and construct full faithful contravariant embedding $\mathscr{C}^{\mathrm{opp}} \xrightarrow{\varrho^\circ} \mathscr{F}un(\mathscr{C}, \mathscr{D})$, which sends an object $A \in \mathrm{Ob}\,\mathscr{C}$ to the covariant functor $\varrho^\circ(A) = h^A$ and sends an arrow $A \xrightarrow{\varphi} B$ to the natural transformation $h^B \xrightarrow{\varrho^\circ(\varphi)} h^A$, whose action over $X \in \mathrm{Ob}\,\mathscr{C}$ is

$$\varrho^\circ(\varphi)_X : h^B(X) = \mathrm{Hom}(B,Y) \xrightarrow{\psi \mapsto \psi \circ \varphi} \mathrm{Hom}(X_1, Y) = h^{X_1}(Y)\,.$$

HINT. Just reverse all the arrows in the previous constructions.

14.6. Defining objects by «universal properties». The functoriality of the representing objects allows to transfer many set-theoretical constructions[2] to an arbitrary category $\mathscr{C}$. Namely, one can

[1] recall that this means coincidence $\mathrm{Hom}_{\mathscr{F}un(\mathscr{C}^{\mathrm{opp}},\mathscr{S}et)}(h_A, h_B) = \mathrm{Hom}_{\mathscr{C}}(A,B)$

[2] such as a direct product of sets e. t. c.

define a result of some set theoretical operation with objects X_i in $\mathscr{C}$ as an object X such that for any Y *a set* $\mathrm{Hom}(Y,X)$ coincides with the result of the original set theoretical operation applied to *sets* $\mathrm{Hom}(Y,X_i)$. In other words, X should represent a functor that takes Y to the result of the set-theoretical operation with $\mathrm{Hom}(Y,X_i)$'s. Of course, this definition is implicit and does not guarantee the existence of X, because the functor in question could be not representable. But if a representing object exists, then it automatically carries some «universal properties» and is unique up to unique isomorphism preserving these properties.

14.6.1. Example: a product $A\times B$, of $A,B\in\mathrm{Ob}\,\mathscr{C}$, is an object representing a functor

$$Y\mapsto \mathrm{Hom}(Y,A)\times\mathrm{Hom}(Y,B)$$

from $\mathscr{C}^{\mathrm{opp}}$ to $\mathscr{S}et$ (as soon it is representable). In more details, for any Y we should have an isomorphism $\beta_Y:\mathrm{Hom}(Y,A\times B)\xrightarrow{\sim}\mathrm{Hom}(Y,A)\times\mathrm{Hom}(Y,B)$ functorial w. r. t. arrows $Y_1\longrightarrow Y_2$. Following the proof from n° 14.5.2, we can put here $Y=A\times B$ and write

$$\beta_{A\times B}(\mathrm{Id}_{A\times B})\in\mathrm{Hom}(A\times B,A)\times\mathrm{Hom}(A\times B,B)$$

as (π_A,π_B) for appropriate arrows $A\xleftarrow{\pi_A}A\times B\xrightarrow{\pi_B}B$.

Exercise 14.5. Show that the triple $A\xleftarrow{\pi_A}A\times B\xrightarrow{\pi_B}B$ satisfies the following universal property: for any two morphisms $A\xleftarrow{\varphi}Y\xrightarrow{\psi}B$ there exists a unique morphism $Y\xrightarrow{\varphi\times\psi}A\times B$ such that $\varphi=\pi_A\circ(\varphi\times\psi)$ and $\psi=\pi_B\circ(\varphi\times\psi)$.

Exercise 14.6. Show that for any other triple $A\xleftarrow{\pi'_A}C\xrightarrow{\pi'_B}B$ satisfying the above universal property there exists a unique isomorphism $\gamma:C\xrightarrow{\sim}A\times B$ such that $\pi_A\circ\gamma=\pi'_A$ and $\pi_B\circ\gamma=\pi'_B$.

14.6.2. Example: a coproduct $A\otimes B$ in an arbitrary category $\mathscr{C}$ is an object corepresenting a functor

$$Y\mapsto\mathrm{Hom}(A,Y)\times\mathrm{Hom}(B,Y)$$

from $\mathscr{C}$ to $\mathscr{S}et$. Reversing arrows in the previous example, we can characterize it by the following universal property: there are two morphisms $A\xrightarrow{i_A}A\otimes B\xleftarrow{i_B}B$ such that for any two arrows $A\xrightarrow{\varphi}Y\xleftarrow{\psi}B$ in $\mathscr{C}$ there exists a unique morphism $A\otimes B\xrightarrow{\varphi\otimes\psi}Y$ such that $\varphi=(\varphi\otimes\psi)\circ i_A$ and $\psi=(\varphi\otimes\psi)\circ i_B$.

Exercise 14.7. Show that such a triple $A\xrightarrow{i_A}A\otimes B\xleftarrow{i_B}B$ (if exists) is unique up unique isomorphism commuting with i-arrows.

Exercise 14.8. Show that if one of two coproducts $A\otimes(B\otimes C)$, $(A\otimes B)\otimes C$ exists, then the other one exists as well and is isomorphic to the first. Prove a similar statement for the products.

14.7. Limits. Two examples above are just very particular cases of much more general construction. Fix some category $\mathscr{N}$ (called *a category of indexes*. A functor $\mathscr{N}\xrightarrow{X}\mathscr{C}$ is nothing but a family of objects $X_\nu\in\mathrm{Ob}\,\mathscr{C}$ indexed by $\nu\in\mathrm{Ob}\,\mathscr{N}$ and morphisms $X_\nu\xrightarrow{\varphi_{\nu\mu}}X_\mu$ indexed by the arrows $i\longrightarrow j$ of $\mathscr{N}$.

For example, if $\mathscr{N}$ is a partially ordered set satisfying the extra condition $\forall\, i,j\ \exists\, k:\ k>i,\ k>j$, then a functor $\mathscr{N}\xrightarrow{X}\mathscr{C}$ is called *a direct spectrum* or *a direct system* of morphisms in $\mathscr{C}$. Dually, a functor $\mathscr{N}\mathrm{opp}\xrightarrow{X}\mathscr{C}$ is called in *an inverse spectrum* or *a inverse system* of morphisms in $\mathscr{C}$.

Further, there is a functor $\mathscr{C}\xrightarrow{X\mapsto\overline{X}}\mathscr{F}un(\mathscr{N},\mathscr{C})$, which attaches to each object $X\in\mathrm{Ob}\,\mathscr{C}$ a constant family $\overline{X}$ (whose $\overline{X}_\nu\equiv X$, $\varphi_{\nu\mu}\equiv\mathrm{Id}_X$) and takes each arrow $X\xrightarrow{\psi}Y$ to the corresponding morphism of constant families $\overline{X}\overset{\psi}{\mapsto}\overline{Y}$.

Given an arbitrary family $\{X_\nu,\varphi_{\nu\mu}\}:\mathscr{N}\longrightarrow\mathscr{C}$, then an object $\varprojlim X_\nu\in\mathscr{C}$ representing a contravariant functor $Y\longmapsto\mathrm{Hom}_{\mathscr{F}un(\mathscr{N},\mathscr{C})}(\overline{Y},X)$ from $\mathscr{C}$ to $\mathscr{S}et$ is called *a projective limit* of the given family. By the definition, there is a functorial in Y isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(Y,\varprojlim X_\nu)=\mathrm{Hom}_{\mathscr{F}un(\mathscr{N},\mathscr{C})}(\overline{Y},X)\ .$$

Applying it to $Y = \varprojlim X_\nu$, we get a natural transformation $\overline{\varprojlim X_\nu} \xrightarrow{\pi} X$ corresponding to

$$\mathrm{Id}_{\varprojlim X_\nu} \in \mathrm{Hom}_{\mathscr{C}}(\varprojlim X_\nu, \varprojlim X_\nu)\ .$$

This transformation is a family of morphisms $\varprojlim X_\nu \xrightarrow{\pi_\nu} X_\nu$ such that $\pi_\mu = \varphi_{\nu\mu}\pi_\nu$ for all arrows $\varphi_{\nu\mu}$ in the family $\{X_\nu\}$. It satisfies the following universal property: for any object $Y \in \mathrm{Ob}\,\mathscr{C}$ equipped with a family of arrows[1] $Y \xrightarrow{\psi_\nu} X_\nu$ such that $\psi_\mu = \varphi_{\nu\mu}\psi_\nu$ there exists a unique morphism $Y \xrightarrow{\alpha} \varprojlim X_\nu$ such that $\psi_\nu = \pi_\nu \circ \alpha\ \forall \nu$.

Exercise 14.9. Show that projective limit is uniquely characterized by this universal property (up to unique isomorphism commuting with π_ν's).

Dually, *an inductive limit* $\varinjlim X_\nu$ corepresents a covariant functor $Y \longmapsto \mathrm{Hom}_{\mathscr{F}un(\mathscr{N},\mathscr{C})}(X, \overline{Y})$.

Exercise 14.10. Show that inductive limit $\varinjlim X_\nu$ is equipped with canonical maps $X_\nu \xrightarrow{i_\nu} \varinjlim X_\nu$ and satisfies the following universal property: given an object $Y \in \mathrm{Ob}\,\mathscr{C}$ with a family of arrows $X_\nu \xrightarrow{\psi_\nu} Y$ such that $\psi_\mu = \psi_\nu\varphi_{\mu\nu}$ (which give a natural transformation $X \overset{\psi}{\longmapsto} \overline{Y}$ in $\mathscr{F}un(\mathscr{N},\mathscr{C})$), then there exists a unique morphism $\varinjlim X_\nu \xrightarrow{\alpha} Y$ such that $\psi_\nu = \alpha \circ \pi_\nu\ \forall \nu$.

Exercise 14.11. Let $\mathscr{N}$ be an arbitrary partially ordered set (considered as a category). Show that any family of $\mathscr{N}$-indexed sets $\mathscr{N} \xrightarrow{X} \mathscr{S}et$ has $\varinjlim X$.

HINT. A right queue of X is a sequence of elements $x_\nu \in X_\nu$ indexed by some $S \subset \mathrm{Ob}\,\mathscr{N}$ such that all $\mu > \nu$ belong to S as soon as $\nu \in S$ and $\varphi_{\mu\nu}(x_\mu) = x_\nu\ \forall \mu, \nu \in S$. Two right queues $\{x_\alpha\}$, $\{y_\beta\}$ are called equivalent, if $\forall\, x_\alpha,\, y_\beta\ \exists \gamma > \alpha, \beta : \varphi_{\alpha\gamma}(x_\alpha) = \varphi_{\beta\gamma}(y_\beta)$. Check that a set of all equivalence classes of right queues satisfies the universal properties defining $\varinjlim X$.

Exercise 14.12. Let $\mathscr{N} = \mathbb{N}$ be the set of all positive integers with the standard order. Find $\varprojlim A_n$ and $\varinjlim A_n$ of abelian groups $A_n = \mathbb{Z}/p^n\mathbb{Z}$ w.r.t. an inverse system of canonical factorizations $\psi_{nm} : \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$ ($\forall\ m < n$) and w.r.t. a direct system of standard inclusions $\varphi_{mn} : \mathbb{Z}/p^m\mathbb{Z} \xhookrightarrow{[1]\mapsto[p^{n-m}]} \mathbb{Z}/p^n\mathbb{Z}$ (again $\forall\, m < n$).

HINT. $\varprojlim A_n = \mathbb{Z}_p$ is the set of all p-adic integers and $\varinjlim A_n \subset \mathbb{Q}/\mathbb{Z}$ consists of (mod $\mathbb{Z}$)-classes of fractions z/p^ℓ whose denominator is a power of p (so called p-rational numbers).

Exercise 14.13. Let $\mathscr{N} = \mathbb{N}$ as above but with the partial ordering prescribed by the divisibility. Find $\varprojlim A_n$ and $\varinjlim A_n$ of $A_n = \mathbb{Z}/n\mathbb{Z}$ w.r.t. an inverse system of factorizations $\psi_{nm} : \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ ($\forall\, m|n$) and w.r.t. a direct system of inclusions $\varphi_{mn} : \mathbb{Z}/m\mathbb{Z} \xhookrightarrow{[1]\mapsto[n/m]} \mathbb{Z}/n\mathbb{Z}$ (again $\forall\, m|n$).

HINT. $\varinjlim A_n = \mathbb{Q}/\mathbb{Z}$ and $\varprojlim A_n = \prod\limits_p \mathbb{Z}_p$ is the product of all rings of p-adic integer numbers.

14.7.1. Example: fibered products (also called Cartesian squares, or coamalgams) are defined in an arbitrary category $\mathscr{C}$ as projective limits w.r.t. the category of indexes $\mathscr{N} = \{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$ (3 objects and 2 non-identical arrows). Any functor $\mathscr{N} \longrightarrow \mathscr{C}$ is a diagram $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ in $\mathscr{C}$. Its projective limit is denoted by $X \underset{B}{\times} Y$ and called *a fibered product of* X, Y *over* B. It comes with the following commutative square (called *a Cartesian square*)

$$\begin{tikzcd} & X \underset{B}{\times} Y \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd} \tag{14-5}$$

[1] that is for any natural transformation $\overline{Y} \overset{\psi}{\longmapsto} X$ in $\mathscr{F}un(\mathscr{N},\mathscr{C})$

which is universal in the following sense: for any other commutative square

$$\begin{tikzcd} & Z \arrow[dl, "\varphi'"'] \arrow[dr, "\psi'"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

there exists a unique morphism $Z \xrightarrow{\varphi' \times \psi'} X \underset{B}{\times} Y$: $\varphi' = \varphi \circ (\varphi' \times \psi')$, $\psi' = \psi \circ (\varphi' \times \psi')$. Upper part of diagram (14-5) is uniquely (up to unique isomorphism commuting with φ, ψ) defined by this universality.

14.7.2. Example: amalgams (also called co-Cartesian squares, or coproducts) are inductive limits w. r. t. the index category $\mathscr{N}$opp $= \{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$. Their expanded definition is obtained from the previous one by reversing the arrows: *an amalgam* of a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is an universal (*co-Cartesian*) commutative square

$$\begin{tikzcd} & X \underset{B}{\otimes} Y & \\ X \arrow[ur, "\varphi"] & & Y \arrow[ul, "\psi"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such that for any other commutative square

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\varphi'"] & & Y \arrow[ul, "\psi'"'] \\ & B \arrow[ul, "\xi"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

there exists a unique morphism $X \underset{B}{\otimes} Y \xrightarrow{\varphi' \otimes \psi'} Z$ satisfying $\varphi' = (\varphi' \otimes \psi') \circ \varphi$, $\psi' = (\varphi' \otimes \psi') \circ \psi$.

14.8. Additive categories. Categories appearing in commutative algebra and geometry typically have extra structures on their morphisms $\mathrm{Hom}(X, Y)$: usually we can add morphisms, form their kernels, images e. t. c. A category $\mathscr{C}$ is called *additive*, if it satisfies the following properties:

- bifunctor $X, Y \longmapsto \mathrm{Hom}(X, Y)$ takes its values in the category of abelian groups $\mathscr{Ab}$ instead of $\mathscr{Set}$, i. e. $\mathrm{Hom}(X, Y)$ is an abelian group $\forall\, X, Y \in \mathrm{Ob}\,\mathscr{C}$ and the composition

$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \xrightarrow{(\varphi,\psi) \mapsto \varphi \circ \psi} \mathrm{Hom}(X, Z)\ ,$$

  is bilinear (or ***distributive***): $(\varphi_1 + \varphi_2) \circ (\psi_1 + \psi_2) = f_1 \circ \psi_1 + f_1 \circ \psi_2 + f_2 \circ \psi_1 + f_2 \circ \psi_2$;

- there is *a zero object* $0 \in \mathrm{Ob}\,\mathscr{C}$ such that $\mathrm{Hom}(0, 0) = 0$ is the zero group;

  Exercise 14.14. Deduce from the previous property that $\mathrm{Hom}(X, 0) = \mathrm{Hom}(0, X) = 0\ \forall\, X \in \mathrm{Ob}\,\mathscr{C}$ and 0 is defined by this property up to unique isomorphism (namely, the zero morphism $0 \xrightarrow{0} 0'$).

- for any pair of objects A, B there exist a diagram[1]:

$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} A \oplus B \underset{\pi_B}{\overset{i_B}{\leftrightarrows}} B \qquad (14\text{-}6)$$

[1] its middle term $A \oplus B$ is called *a direct sum* of A, B and all the diagram is called *a splitted exact triple*

such that $\pi_b \circ i_A = 0$, $\pi_A \circ i_B = 0$, $\pi_A \circ i_A = \mathrm{Id}_A$, $\pi_B \circ i_B = \mathrm{Id}_B$ and $i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{Id}_{A \oplus B}$.

Exercise 14.15. Show that $A \oplus B$ is defined by this property up to unique isomorphism commuting with i's and π's.

One can emulate all natural constructions known for abelian groups in a context of an arbitrary additive category $\mathscr{C}$. For example, define *a kernel* of an arrow $A \xrightarrow{\varphi} B$ in $\mathscr{C}$ as an object representing a functor

$$C \longmapsto \ker\Big(\mathrm{Hom}(C, A) \xrightarrow{\gamma \mapsto \varphi \circ \gamma} \mathrm{Hom}(C, B)\Big)$$

from $\mathscr{C}$ to $\mathscr{Ab}$. If exists, the representing object $\ker(\varphi)$ comes with canonical map[1] $\ker(\varphi) \xrightarrow{\varkappa} A$ satisfying $\varphi \circ \varkappa = 0$ and the following universality: for any arrow $C \xrightarrow{\gamma} A$ such that $\varphi \circ \gamma = 0$ there exists a unique morphism $C \xrightarrow{\psi} \ker(\varphi)$ such that $\varkappa \circ \psi = \gamma$. This property fixes the kernel up to unique isomorphism commuting with $\varkappa$. Reversing arrows, we define *a cokernel* of $A \xrightarrow{\varphi} B$ as a universal morphism $B \xrightarrow{\chi} \mathrm{coker}\,(\varphi)$ such that $\chi \circ \varphi = 0$ and for any arrow $B \xrightarrow{\gamma} C$ such that $\gamma \circ \varphi = 0$ there exists a unique morphism $\mathrm{coker}\,(\varphi) \xrightarrow{\psi} C$ such that $\psi \circ \chi = \gamma$. Again, $\mathrm{coker}\,(\varphi)$ is uniquely defined by this property (up to unique isomorphism commuting with χ).

Exercise 14.16. Show that in the direct sum diagram (14-6) the arrow $A \xrightarrow{i_A} A \oplus B$ gives the kernel of the arrow $A \oplus B \xrightarrow{\pi_B} B$ and the arrow $B \xrightarrow{i_B} A \oplus B$ gives the kernel of the arrow $A \oplus B \xrightarrow{\pi_A} A$.

From the main theorem about homomorphisms of groups we expect two ways in which *an image* of arrow $A \xrightarrow{\varphi} B$ could be defined. Namely, $\mathrm{im}\,\varphi$ should be isomorphic to both: the kernel of $B \xrightarrow{\chi} \mathrm{coker}\,(\varphi)$ and the cokernel of $\ker(\varphi) \xrightarrow{\varkappa} A$.

Exercise 14.17. Let $\mathscr{C}$ be an arbitrary additive category and $A \xrightarrow{\varphi} B$ be any arrow in $\mathscr{C}$ such that both $\ker\varphi$ and $\mathrm{coker}\,\varphi$ exist. Show that there is a canonical arrow $\mathrm{coker}\Big(\ker(\varphi) \xrightarrow{\varkappa} A\Big) \longrightarrow \ker\Big(B \xrightarrow{\chi} \mathrm{coker}\,(\varphi)\Big)$.

14.9. Abelian categories. An additive category $\mathscr{C}$ is called *abelian*, if it satisfies

- each arrow $A \xrightarrow{\varphi} B$ has kernel $\ker(\varphi)$, cokernel $\mathrm{coker}\,(\varphi)$ and is decomposed as

$$\begin{tikzcd} & \mathrm{im}\,(\varphi) \arrow[dr, "\beta"] & \\ A \arrow[ur, "\alpha"] \arrow[rr, "\varphi"] & & B \end{tikzcd}$$

  where $\mathrm{im}\,\varphi \simeq \mathrm{coker}\Big(\ker(\varphi) \xrightarrow{\varkappa} A\Big) \simeq \ker\Big(B \xrightarrow{\chi} \mathrm{coker}\,(\varphi)\Big)$.

A morphism φ in abelian category is called *surjective* (or *an epimorphism*), if $\mathrm{coker}\,\varphi = 0$. If $\ker\varphi = 0$, then φ is called *injective* (or *an monomorphism*).

Exercise 14.18. Show that in abelian category:

a) $\ker(\varphi) \xrightarrow{\varkappa} A$ is injective and $B \xrightarrow{\chi} \mathrm{coker}\,(\varphi)$ is surjective for any arrow $A \xrightarrow{\varphi} B$;

b) φ is an isomorphism iff it is simultaneously surjective and injective.

Exercise 14.19. Check that in any additive category all squares

$$\begin{tikzcd} A \oplus B \arrow[r, shift left, "\pi_B"] \arrow[d, shift left, "\pi_A"] & B \arrow[l, shift left, "i_B"] \arrow[d, shift left, "0"] \\ A \arrow[u, shift left, "i_A"] \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"] \arrow[u, shift left, "0"] \end{tikzcd}$$

[1] that is the image of $\mathrm{Id}_{\ker\varphi}$ under the canonical identification

$$\mathrm{Hom}(\ker\varphi, \ker\varphi) \simeq \ker\Big(\mathrm{Hom}(\ker\varphi, A) \xrightarrow{\gamma \mapsto \varphi \circ \gamma} \mathrm{Hom}(\ker\varphi, B)\Big)$$

(coming from (14-6)) are simultaneously Cartesian and co-Cartesian.

14.9.1. CLAIM. *In any abelian category there exist all fibered products and amalgams.*

Proof. To complete an arbitrary triple $X \xrightarrow{\xi} B \xleftarrow{\eta} Y$ to Cartesian square, write $K \xrightarrow{\varkappa} X \oplus Y$ for the kernel of morphism $\delta = \xi \circ \pi_X - \eta \circ \pi_Y \, : \, X \oplus Y \longrightarrow B$. Then a square

$$\begin{tikzcd} & K \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[dr, "\xi"'] & & Y \arrow[dl, "\eta"] \\ & B & \end{tikzcd}$$

where $\varphi = \pi_X \circ \varkappa$, $\psi = \pi_Y \circ \varkappa$ is commutative (because $\xi\varphi - \eta\psi = \delta\varkappa = 0$) and universal (because for any other triple $X \xleftarrow{\varphi'} Z \xrightarrow{\psi'} Y$ such that $\xi\varphi' = \eta\psi'$ only the canonical map[1] $\zeta = \varphi' \oplus \psi' : Z \longrightarrow X \oplus Y$ satisfies $\pi_X\zeta = \varphi'$, $\pi_Y\zeta = \psi'$ and can be lifted to an arrow $Z \xrightarrow{\zeta'} K$, since of $\delta\zeta = \xi\varphi' - \eta\psi' = 0$). $\square$

Exercise 14.20. Show that a diagram $X \xleftarrow{\xi} B \xrightarrow{\eta} Y$ is completed to co-Cartesian square by a cokernel $X \oplus Y \xrightarrow{\chi} Q$ of a morphism $\delta = i_X \circ \xi - i_Y \circ \eta \, : \, B \longrightarrow X \oplus Y$.

Exercise 14.21. Show that for any fibered product (14-5) in abelian category:

a) ξ is surjective $\Rightarrow$ ψ is surjective;

b) $K \xrightarrow{\varkappa} X \underset{B}{\times} Y$ is the kernel of $\varphi \Rightarrow K \xrightarrow{\psi \circ \varkappa} Y$ is the kernel of η.

[1] predicted by ex. 14.19

§15. Vector bundles.

15.1. Fibered products. Given two families $Y_1 \xrightarrow{\pi_1} X$, $Y_2 \xrightarrow{\pi_2} X$ of algebraic manifolds over X (comp. with n° 12.6.3), then their *fibered product* over X is

$$Y_1 \underset{X}{\times} Y_2 \stackrel{\text{def}}{=} \{(y_1, y_2) \in Y_1 \times Y_2 \mid \varphi_1(y_1) = \varphi_2(y_2)\} .$$

In fact, this product comes with a natural structure of a geometric scheme. Namely, if $X = \operatorname{Spec_m} K$, $Y_i = \operatorname{Spec_m} A_i$, where K, A_1, A_2 are (finitely generated reduced) $\Bbbk$-algebras, then the pull-backs $K \xrightarrow{\pi_i^*} A_i$ equip A_i with K-algebra structure and $Y_1 \underset{X}{\times} Y_2 = \operatorname{Spec_m} A_1 \underset{K}{\otimes} A_2$, where $A_1 \underset{K}{\otimes} A_2$ is *the tensor product* of K-algebras A_i over K, that is the quotient algebra of $A_1 \otimes A_2$ by an ideal spanned by all differences $(\varkappa a_1) \otimes a_2 - a_1 \otimes (\varkappa a_2)$, where $\varkappa \in K$, $a_i \in A_i$.

Exercise 15.1. Write $A_i \xrightarrow{\alpha_i^*} A_1 \underset{K}{\otimes} A_2$ for two K-algebra homomorphisms sending $a_1 \in A_1$ and $a_2 \in A_2$ to $a_1 \otimes 1$ and $1 \otimes a_2$ respectively. Show that for any K-algebra B and any two homomorphisms of K-algebras $A_i \xrightarrow{g_i^*} B$ there exists a unique homomorphism of K-algebras $A_1 \underset{K}{\otimes} A_2 \xrightarrow{g_1^* \otimes g_2^*} B$ such that $g_i^* = (g_1^* \otimes g_2^*) \circ \alpha_i^*$ for both $i = 1, 2$. Show also that this universality determinate the triple $(\alpha_1^*, \alpha_2^*, A_1 \underset{K}{\otimes} A_2)$ uniquely up to unique isomorphism commuting with α_i^*'s.

HINT. This is completely similar to ex. 11.3.

So, $Y_1 \underset{X}{\times} Y_2 \subset Y_1 \times Y_2$ is a closed submanifold equipped with two projections $Y_1 \underset{X}{\times} Y_2 \xrightarrow{\alpha_i} Y_i$ and satisfying the following universal property[1]: for any family $Z \xrightarrow{f} X$ and any two morphisms of X-families $Z \xrightarrow{g_i}$ there exists a unique morphism of X-families $Z \xrightarrow{g_1 \times g_2} Y_1 \underset{X}{\times} Y_2$ such that $g_i = \alpha_i \circ (g_1 \times g_2)$, $i = 1, 2$.

It is very important that $\Bbbk$-algebra $A_1 \underset{K}{\otimes} A_2$ can be *non reduced* even if all three algebras in question are reduced (see n° 15.1.2 below). In this case $Y_1 \underset{X}{\times} Y_2$ is always considered as *geometric scheme* canonically equipped with the structure algebra $\Bbbk[Y_1] \underset{\Bbbk[X]}{\otimes} \Bbbk[Y_2]$.

15.1.1. Example: base change. Any family $Y \xrightarrow{\pi} X$ can be lifted along any morphism[2] $X' \xrightarrow{f} X$ to the family $Y \underset{X}{\times} X' \xrightarrow{f^*(\pi)} X'$ fitting into commutative diagram

$$\begin{tikzcd}
Y \underset{X}{\times} X' \arrow[r] \arrow[d, "f^*(\pi)"'] & Y \arrow[d, "\pi"] \\
X' \arrow[r, "f"] & X
\end{tikzcd}$$

This procedure is called *a basis change*. Algebraically, it is known as *extension of scalars*. For example, given $\mathbb{R}$-algebra (or just a vector space) V, then its complexification is nothing but $\mathbb{C} \underset{\mathbb{R}}{\otimes} V$.

15.1.2. Example: scheme restrictions and scheme preimages. Given a closed embedding $Z \overset{\varphi}{\hookrightarrow} X$ and an arbitrary family (i. e. a regular map) $Y \xrightarrow{f} X$, then the basis change $Y \underset{X}{\times} Z \xrightarrow{f^*(\varphi)} Z$, of f along φ, is called *a scheme restriction* of the family Y onto the closed submanifold Z and the basis change $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$, of f along φ, is called *a scheme preimage* of the closed submanifold $Z \subset X$ under the morphism $Y \xrightarrow{f} X$. If X is affine and Z is given by an ideal $I \subset \Bbbk[X]$, then geometrically $Z \underset{X}{\times} Y \overset{f^*(\varphi)}{\hookrightarrow} Y$ is a closed embedding of

[1] note that it is stronger than the set theoretical definition of $Y_1 \underset{X}{\times} Y_2$ given before and obtained by the specialization of the universality in question to $Z = \pi_1^{-1}(x) \times \pi_2^{-1}(x)$ for $x \in X$

[2] mathematically, «family» and «morphism» mean the same; we use different words just to outline the different roles of these maps but it is extremely important that these roles are completely symmetric

$\mathrm{Spec_m}\left(\Bbbk[X]/I)\underset{\Bbbk[X]}{\otimes}\Bbbk[Y]\right)_{\mathrm{red}}$ into Y, which identifies $f^{-1}(Z)$ with the zero set of ideal $(f^*)^{-1}(I)$. But in general the structure algebra $\Bbbk[X]/I)\underset{\Bbbk[X]}{\otimes}\Bbbk[Y]$ is non reduced.

For example, consider a scheme preimage of cuspidal cubic $Z \subset \mathbb{A}_2$ given by equation $y^2 = x^3$ along the map $\mathbb{A}_1 \xrightarrow{t\mapsto(t,t^2)} \mathbb{A}_2$ whose image is the parabola $y = x^2$. It consists of two points $t = 0$ and $t = 1$ but is equipped with non reduced structure algebra[1] $\Bbbk[t]\underset{\Bbbk[x,y]}{\otimes}\big(\Bbbk[x,y]/(y^2-x^3)\big) = \Bbbk[t]/(t^4-t^3)$, which keeps the local intersection multiplicities.

15.2. Algebraic vector bundle over an algebraic manifold X is an algebraic family of vector spaces over X, i. e. a regular map of algebraic manifolds $E \xrightarrow{\pi} X$ whose fiber $\pi^{-1}(x)$ over any $x \in X$ has a structure of a vector space over $\Bbbk$ and this structure algebraically depends on x in a sense that fiberwise operations[2]:

- pick up the zero: $X \xrightarrow{x\mapsto[0]_x} E$
- add vectors: $E\underset{X}{\times}E \xrightarrow{([u]_x,[v]_x)\mapsto[u+v]_x} E$
- multiply vectors by constants $\left(X\times\mathbb{A}^1\right)\underset{X}{\times}E \xrightarrow{([\lambda]_x,[v]_x)\mapsto[\lambda v]_x} E$

are the regular morphisms of algebraic manifolds and commute with the projections onto X.

Two vector bundles $E_1 \xrightarrow{\pi_1} X$, $E_2 \xrightarrow{\pi_2} X$ are called *isomorphic*, if there is an isomorphism of algebraic varieties $E_1 \xrightarrow{\varphi} E_2$ such that $\pi_2\circ\varphi = \pi_1$ and $\forall\, x\in X$ the restriction $\pi_1^{-1}(x) \xrightarrow{\varphi|_{\pi_1^{-1}(x)}} \pi_2^{-1}(x)$ is linear isomorphism of vector spaces.

A vector bundle is called *trivial* of rank d, if it is isomorphic to the direct product $X\times\mathbb{A}_d$ with the standard vector space structure on $\mathbb{A}_d = k^{\oplus d}$, which does not depend on $x \in X$.

A regular map $X \xrightarrow{s} E$ is called *a section*, if $\pi\circ s = \mathrm{Id}_X$, i. e. $s(x) \in \pi^{-1}(x)\ \forall x$. Each vector bundle has canonical *zero section*, which takes the zero at each fiber. A vector bundle $E \xrightarrow{\pi} X$ is trivial of rank d iff there are d regular sections $X \xrightarrow{s_i} E$ such that $\{s_1(x), \dots, s_d(x)\}$ form a basis of $\pi^{-1}(x)$ $\forall\, x\in X$. Indeed, the fiberwise coordinate functions on E w. r. t. these basic vectors give the required isomorphism $E \xrightarrow{\sim} X\times\mathbb{A}_d$.

15.3. Locally trivial vector bundle of rank d is a vector bundle $E \xrightarrow{\pi} X$ such that any $x \in X$ has an open neighborhood U such that the restricted bundle $\pi^{-1}(U) \longrightarrow U$ is trivial of rank d, i. e. has d basic sections $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) : U \longrightarrow \pi^{-1}(U)$. If there are two such trivializations $(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)})$ and $(s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)})$ defined, respectively, over some open U, V, then over each $x \in U\cap V$ these two basises are expressed through each other as[3]

$$(s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)}) = (s_1^{(V)}, s_2^{(V)}, \dots, s_d^{(V)})\cdot\varphi_{VU}\ ,$$

where $\varphi_{VU} = \varphi_{VU}(x)$ is a non degenerate $d\times d$ - matrix whose entries are regular functions on $U\cap V$. So, we get the regular maps $U\cap V \xrightarrow{\varphi_{VU}} \mathrm{GL}_d(k)$ called *transition functions* between two given trivializations. They clearly satisfy the conditions

$$\varphi_{UV} = \varphi_{VU}^{-1}\ ,\quad \varphi_{VU}\varphi_{UW} = \varphi_{VW} \tag{15-1}$$

(the latter hold over any triple intersection $U\cap V\cap W$). If we change local basis over each open set U by some other

$$(\widetilde{s}_1^{(U)}, \widetilde{s}_2^{(U)}, \dots, \widetilde{s}_d^{(U)}) = (s_1^{(U)}, s_2^{(U)}, \dots, s_d^{(U)})\cdot\psi_U\ ,$$

[1] note that both x and y act on $\Bbbk[t]$ as t and t^2 respectively

[2] we write $[v]_x$ for elements in the fiber $\pi^{-1}(x)$ over $x \in X$

[3] i. e. i-th column of φ_{VU} contains the coordinates of $s_i^{(U)}(x)$ w. r. t. the basis $\{s_1^{(V)}(x), \dots, s_d^{(V)}(x)\}$

where $\psi_U = \psi_U(x)$ is any non degenerate $d \times d$ matrix whose entries are regular functions on the whole of U, then the transition functions also will be changed by $\widetilde{\varphi}_{VU} = \psi_V^{-1}\varphi_{VU}\psi_U$.

15.4. $\mathrm{GL}_d(k)$-valued Čhech's 1-cocycle on X associated with an open covering $X = \cup U_\nu$ is a series of regular maps $U_\alpha \cap U_\beta \xrightarrow{\varphi_{\alpha\beta}} \mathrm{GL}_d(k)$ defined for any ordered pair of indexes (α, β) and such that $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ over $U_\alpha \cap U_\beta$ for all α, β and $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ over $U_\alpha \cap U_\beta \cap U_\gamma$ for all α, β, γ. Any such cocycle produces a cocycle associated with any finer[1] covering inscribed in the initial one (just restrict $\varphi_{\alpha\beta}$ onto the smaller open sets). Two Chech 1-cocycles are called *equivalent* or *(co)homologous*, if there exist some common refinement $X = \cup U_\nu$ of their initial open coverings and some regular maps $U_\nu \xrightarrow{\psi_\nu} \mathrm{GL}_d(k)$ such that the functions $\varphi_{\alpha\beta}$, $\widetilde{\varphi}_{\alpha\beta}$, induced by these cocycles on the refinement, satisfy the equation $\widetilde{\varphi}_{\alpha\beta} = \psi_\alpha^{-1}\varphi_{\alpha\beta}\psi_\beta$ over each $U_\alpha \cap U_\beta$. An equivalence class of Čhech 1-cocycles is called *a first Čech cohomology*. The set of these cohomologies is denoted by $H^1(X, \mathrm{GL}_d(k))$.

15.4.1. CLAIM. *Isomorphism classes of locally trivial vector bundles of rank d are in 1–1 correspondence with the first Čech cohomologies $\{\varphi_{\alpha\beta}\} \in H^1(X, \mathrm{GL}_d(k))$.*

Proof. Given a Čech cocycle $\varphi_{\alpha\beta}$, construct E as a manifold whose atlas consists of affine charts are $U_\alpha \times \mathbb{A}_d$ glued along $(U_\alpha \cap U_\beta) \times \mathbb{A}_d$ by the rule

$$U_\alpha \times \mathbb{A}_d \ni (x, v) \longleftrightarrow (x, \varphi_{\alpha\beta}(x) \cdot v) \in U_\beta \times \mathbb{A}_d \ ,$$

where $v \in \mathbb{A}_d$ is a column vector. The cocycle conditions imply that these chard form an atlas and linearity of $\varphi(\alpha\beta)(x)$ for each x implies that the vector space structures of fibers are correctly glued together. Conversely, we have seen in the previous section that for a given vector bundle the transition functions between its trivializations form a Čech cocycle, which is changed by a homologous one under a changing of the trivialization or (what is the same) under a a fiberwise linear isomorphism of the bundle. □

15.4.2. Example: a tautological vector bundle $S \twoheadrightarrow \mathbb{P}(V)$ is rank 1 vector subbundle of the trivial bundle $\mathbb{P}(V) \times V$ such that a fiber of S over $v \in \mathbb{P}(V)$ is 1-dimensional subspace of V spanned by v. Over any affine chart $U_\alpha = \{v \in \mathbb{P}(V) \mid \alpha(v) \neq 0\}$, where $\alpha \in V^*$, it can be trivialized by the section $s^{(\alpha)}(v) = (v, v/\alpha(v)) \in \mathbb{P}(V) \times V$, which is a well defined regular function $U_\alpha \xrightarrow{s^{(\alpha)}} S \subset \mathbb{P}(V) \times V$. Since $s^{(\alpha)}(v) = s^{(\beta)}(v) \cdot (\beta(v)/\alpha(v))$ over each $v \in U_\alpha \cap U_\beta$, the transition functions between these trivializations are $\varphi_{\beta\alpha}(v) = \beta(v)/\alpha(v)$, which are well defined regular maps $U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_1(k) = k^*$.

15.4.3. Example: a tautological vector bundle $S \twoheadrightarrow \mathrm{Gr}(m, V)$ over the Grassmannian, whose points are m-dimensional subspaces $W \subset V$, is a rank m vector subbundle $S \subset \mathrm{Gr}(m, V) \times V$ whose fiber over $W \in \mathrm{Gr}(m, V)$ is the m-dimensional subspace $W \subset V$ itself. If we fix a basis $\{e_1, e_2, \dots, e_n\}$ in V and for each

$$I = (i_1 < i_2 < \cdots < i_m) \subset (1, 2, \dots, n)$$

consider the standard affine chard $U_I \subset \mathrm{Gr}(m, V)$, which consists of all W projected isomorphically onto the linear span of $\{e_{i1}, e_{i2}, \dots, e_{im}\}$, then S is trivialized over U_I by m sections $s_\nu^{(I)}(W) \subset W$ that form a unique basis of W such that the coordinates of the basic vectors form $m \times n$ matrix $M_I(W)$ containing the identity $m \times m$ - submatrix in the rows I. Since for any $W \in U_I \cap U_J$ we have $M_I(W) = M_J(W) \cdot \varphi_{JI}(W)$, where $\varphi_{JI}(W)$ is the inverse matrix for the $m \times m$ submatrix of M_J situated in the rows I, the transition functions between two trivializations $s_\nu^{(I)}(W)$ and $s_\nu^{(J)}(W)$ are given by the maps $W \longmapsto \varphi_{JI}(W) \in \mathrm{GL}_m(k)$. Clearly, these are regular maps well defined everywhere in $U_I \cap U_J$.

15.5. Linear constructions with vector bundles. Given two locally trivial vector bundles E, F of ranks r, s presented by Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$ over the same open covering $X = \cup U_\nu$, one can form their fiberwise direct sum $E \oplus F$, which has rank $r + s$ and Čech cocycle $\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}$ (direct sum of linear operators), and fiberwise tensor product $E \otimes F$, which has rank rs and Čech cocycle $\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}$ (tensor product of linear operators). Similarly one can make other tensor constructions, say fiberwise exterior or symmetric powers $\Lambda^m E$, $S^m E$ of a given locally trivial vector bundle E e. t. c.

15.6. Pull back. Given a regular map $X \xrightarrow{f} Y$, then any vector bundle $E \twoheadrightarrow Y$ induces a vector bundle $f^*(E) \overset{\text{def}}{=} X \underset{Y}{\times} E \twoheadrightarrow X$ over X called *a pull back* of E along f. For locally trivial E presented

[1] a covering $X = \cup W_\nu$ is called *finer* than a covering $X = \cup U_\nu$, if $\forall \nu\ \exists \mu\ :\ W_\nu \subset U_\nu$)

by Čech cocycle $\varphi_{\alpha\beta}$ over some open covering $Y = \cup U_\nu$, the pull back f^*E is also locally trivial bundle presented by $f^*(\varphi_{\alpha\beta}) = \varphi_{\alpha\beta}\circ f$ over the induced open covering $X = \cup f^{-1}(U_\nu)$.

Exercise 15.2. Let $\mathrm{Gr}(m,V) \xhookrightarrow{p} \mathbb{P}(\Lambda^m V)$ be the Plücker embedding. Check that the pull back $p^*S_{\mathbb{P}}$ of the tautological line bundle on $\mathbb{P}(\Lambda^m V)$ is the maximal exterior power $\Lambda^m S_{\mathrm{Gr}}$ of the tautological line bundle on $\mathrm{Gr}(m,V)$.

15.7. Picard group. Isomorphism classes of locally trivial algebraic vector bundles of rank one on X carry a natural structure of abelian group w.r.t. the tensor multiplication. This group is called *the Picard group* and is denoted $\mathrm{Pic}\,(X)$. Given two line bundles L, K with Čech cocycles $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$, which are $\Bbbk^*$ - valued functions on $U_\alpha \cap U_\beta$ in this case, then their sum in $\mathrm{Pic}\,(X)$ equals to the line bundle $E\otimes K$ with the Čech cocycle $\varphi_{\alpha\beta}\cdot\psi_{\alpha\beta}$. The zero element of $\mathrm{Pic}\,(X)$ is the trivial line bundle $I = X\times\mathbb{A}_1$. The opposite element for a line bundle L with Čech cocycle φ_{ij} is the dual bundle $L^* = \mathrm{Hom}(L,I)$ wjth Čech cocycle equals $\varphi^*_{ij} = 1/\varphi_{ij}$.

15.7.1. THEOREM. *If X is affine and $\Bbbk[X]$ is factorial, then* $\mathrm{Pic}\,(X) = 0$.

Proof. Given line bundle L, we can always chose a trivializing covering $X = \cup U_\alpha$ such that $U_\alpha = \mathscr{D}(f_\alpha)$ for some finite collection $f_1, f_2, \dots, f_n \in \Bbbk[X]$. Let us fix a trivializing section s_α over each U_α and consider corresponding transition functions $\varphi_{\beta\alpha} = s_\beta/s_\alpha$, which are nowhere vanishing elements of $\mathscr{O}_X(U_\alpha\cap U_\beta) = \Bbbk[X][1/(f_\alpha f_\beta)]$, i.e. have a form $f_\alpha^r f_\beta^s$ for some $r,s\in\mathbb{Z}$. Consider some irreducible element $q\in\Bbbk[X]$ and $m_{\beta\alpha}\in\mathbb{Z}$ for a power of q in the prime decomposition of $\varphi_{\beta\alpha}$. If at least one of these powers is not zero, we can split all f_α's into two nonempty subsets: φ_β's, which are divisible by q, and φ_γ's, which are not. Then, for each β the power $m_{\gamma\beta}$ does not depend on γ, because q must disappear in $\varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_1\beta}/\varphi_{\gamma_2\beta}$. Let us write m_β for this power and change all sections s_β by $s'_\beta = q^{m_\beta}\cdot s_\beta$ (this leads to a new basic section, because $Z_q\subset Z_{f_\beta}$). After that q, clearly, disappears in all $\varphi_{\gamma\beta}$ as well as in all $\varphi_{\beta_1\beta_2} = \varphi_{\gamma\beta_2}/\varphi_{\gamma\beta_1}$. Since the set of all q's having some non zero $m_{\alpha\beta}$ is exhausted by a finite number of irreducible divisors of f_α's, after a number of such the replacements we come to transition functions that have no irreducible factors, i.e. are non zero constants. Rescaling all but one basic sections, we come to a global trivialization for L. □

15.7.2. COROLLARY. $\mathrm{Pic}\,(\mathbb{A}_n) = 0$. □

15.7.3. PROPOSITION. $\mathrm{Pic}\,(\mathbb{P}_n) = \mathbb{Z}$ *is spanned by the tautological vector bundle* S.

Proof. By n° 15.7.2, any line bundle L can be trivialized over the standard affine chart U_{x_i} by some local nowhere vanishing section s_i. Let us write $t_\nu^{(i)}$, $\nu\neq i$, for the restrictions of linear forms x_ν onto affine hyperplane $x_i = 0$ in $\mathbb{A}_{n+1}$ and use them as affine coordinates on U_{x_i}. The transition function $\varphi_{ij} = s_i/s_j \in k(U_{x_i})$ is a rational function of $t_\nu^{(i)}$ such that its numerator and denominator do not vanish anywhere in U_{x_i} except for $Z_{t_j^{(i)}}$. Hence, $\varphi_{ij} = (t_j^{(i)})^{d_{ij}}$. Since $t_k^{(j)} = x_k/x_j = (x_k/x_i) : (x_j/x_i) = t_k^{(i)}/t_j^{(i)}$, the cocycle conditions $\varphi_{ij} = 1/\varphi_{ji}$ and $\varphi_{jk} = \varphi_{ik}/\varphi_{ij}$ force $d_{ij} = -d_{ji} = d$ with the same d for all i, j. On the other side, for any $d\in\mathbb{Z}$ the functions $\varphi_{ij} = (t_j^{(i)})^d = (x_j/x_i)^d$ form Čech 1-cocycle, i.e. define a line bundle, which we will denote by $\mathscr{O}(-d)$.

Exercise 15.3. Check that $\mathscr{O}(-d) = S^{\otimes d}$.

So, it remains to show that all $\mathscr{O}(d)$ are pairwise non isomorphic. To this aim let us describe spaces $\Gamma(X,\mathscr{O}(d))$, of their regular global sections. A local section defined everywhere on U_{x_0} has a form $s = f(t_1^{(0)}, t_2^{(0)}, \dots, t_n^{(0)})\cdot s_0$, where f is an arbitrary polynomial of n variables. Rewriting s in terms of chart U_{x_i} we get

$$s = f\left(\left(t_1^{(i)}/t_0^{(i)}\right), \dots, \left(t_n^{(i)}/t_0^{(i)}\right) \right)\cdot \left(t_0^{(i)}\right)^d \cdot s_i\,.$$

So, s is extended onto U_{x_i} iff $\deg f \leqslant d$. Hence, $\dim\Gamma(X,\mathscr{O}(d)) = 0$ for $d<0$ and

$$\dim\Gamma(X,\mathscr{O}(d)) = \binom{n+d}{d} \quad \text{for } d\geqslant 0\,.$$

In particular, all positive $\mathscr{O}(d)$ are mutually different and non isomorphic to negative. Since $\mathscr{O}(-d) = \mathscr{O}(d)^*$, the bundles $\mathscr{O}(d)$ with negative d are pairwise different as well. □

15.8. Sheaves of sections. Let $E \xrightarrow{\pi} X$ be a locally trivial vector bundle. Then for any open $U\subset X$ all regular local sections $U \xhookrightarrow{s} \pi^{-1}(U)\subset E$ form a module over an algebra of local regular functions

$\mathscr{O}_X(U)$. This module is denoted by $\Gamma(U,E)$ or $E(U)$. The correspondence $U \longmapsto \Gamma(U,E)$ is called *a sheaf of local sections* of the vector bundle E. Regardless of an evident ambiguity, it is usually denoted by the same letter E. Since the bundle E is locally trivial, the sheaf E is *locally free*, i.e. each point $x \in X$ has an open neighborhood $U \ni x$ such that $\Gamma(U,E)$ is finitely generated free $\mathscr{O}_X(U)$ - module of rank $\operatorname{rk} E$. Indeed, any collection of trivializing sections for E over U gives a free basis for $\Gamma(U,E)$ over $\mathscr{O}_X(U)$.

15.8.1. LEMMA. *Let E be a vector bundle over an affine irreducible variety X and $P_E = \Gamma(E,X)$ be $\Bbbk[X]$-module of its global sections. Then P_E is finitely generated and torsion free*[1]*. For any $g \in \Bbbk[X]$ module of local sections $\Gamma(\mathscr{D}(g),E)$ coincides with $\Bbbk[X][g^{-1}] \underset{\Bbbk[X]}{\otimes} P_E$, which is the module of fractions*[2] *s/g^m, where $s \in P_E$, $m \in \mathbb{Z}$.*

Proof. Let E be trivialized over some principal open covering $X = \cup \mathscr{D}(f_\nu)$ by local sections

$$s_1^{(\nu)}, s_2^{(\nu)}, \dots, s_r^{(\nu)} \in \Gamma(\mathscr{D}(f_\nu), E) .$$

Then the restriction of any section $s \in \Gamma(\mathscr{D}(g),E)$ onto $\mathscr{D}(g) \cap \mathscr{D}(f_\nu) = \mathscr{D}(gf_\nu)$ can be written as

$$s|_{\mathscr{D}(gf_\nu)} = \sum_{i=1}^{r} \frac{h_i}{(gf_\nu)^{m_\nu}} \cdot s_i^{(\nu)}|_{\mathscr{D}(gf_\nu)}$$

So, $\widetilde{s} = g^{\max m_\nu} \cdot s$ is extended onto each $\mathscr{D}(f_\nu)$, that is to a global section of E, and $s = \widetilde{s}/g^m$ as required in the last assertion. To prove the first assertion, write $s_i^{(\nu)}$ as $s_i^{(\nu)} = \widetilde{s}_i^{(\nu)}/f_\nu^{m_{i\nu}}$, where $\widetilde{s}_i^{(\nu)} \in P_E$ are global sections. Then $\widetilde{s}_i^{(\nu)}$ generate P_E over $\Bbbk[X]$. Indeed, for any $s \in P_E$ and any ν we can write: $s|_{\mathscr{D}(f_\nu} = \dfrac{1}{f_\nu^m} \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ for some $g_i^{(\nu)} \in \Bbbk[X]$ and $m \in \mathbb{N}$. Hence, $f_\nu^m \cdot s = \sum\limits_i g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$ is a $\Bbbk[X]$-linear combination of $\widetilde{s}_i^{(\nu)}$'s. On the other hand, we can write $1 = \sum\limits_\nu h_\nu f_\nu^m$, because f_ν's have no common zeros. So, $s = \sum_\nu h_\nu f_\nu^m \cdot s = \sum_{\nu i} h_\nu g_i^{(\nu)} \cdot \widetilde{s}_i^{(\nu)}$. Absence of torsion is evident. □

15.8.2. COROLLARY. *Under the previous claim conditions, E is trivial iff P_E is free.*

Proof. If $s_1, s_2, \dots, s_r$ form the basis of P_E, then, by the claim, their restrictions onto each $\mathscr{D}(f)$ form the basis of $\Gamma(\mathscr{D}(f),E)$ over $\mathscr{O}_X(\mathscr{D}(f))$. In particular, r coincides with the number of local trivializing sections, i.e. with the rank of E. Moreover, $s_1, s_2, \dots, s_r$ form a basis in each fiber. Indeed, if some fiber contains a vector lying outside a linear span of s_i's, then a local section drawn through this vector can not be expressed as $\mathscr{O}_X$-linear combination of s_i's. □

15.8.3. COROLLARY. *Each algebraic locally trivial vector bundle over $\mathbb{A}_1$ is trivial.*

Proof. P_E is free, because any finitely generated torsion free $\Bbbk[t]$-module is free[3]. □

Exercise 15.4. Show that any nowhere vanishing regular section of an algebraic vector bundle over $\mathbb{A}_1$ can be included in some system of global regular sections forming a basis in each fiber.

15.8.4. THEOREM (BIRKHOFF–GROTHENDIECK). *Each locally trivial algebraic vector bundle of rank r over $\mathbb{P}_1$ is a direct sum of line bundles $\mathscr{O}_{\mathbb{P}_1}(d_i)$ for appropriate $d_1, d_2, \dots, d_r \in \mathbb{Z}$.*

Proof. Write t for affine coordinate on $\mathbb{A}_1 = \mathbb{P}_1 \setminus \{\infty\}$ and consider two trivializations for a given vector bundle E

$$(e_1^0, e_2^0, \dots, e_r^0) , \quad (e_1^\infty, e_2^\infty, \dots, e_r^\infty)$$

[1] A-module M is called *torsion free*, if $am = 0 \Rightarrow a = 0$ or $m = 0$ for $a \in A$, $m \in M$

[2] also called *a localization* of P_E w.r.t. multiplicative set $\{f^k\}$

[3] The same is true for any principal ideal domain. For proof, present the module M in question as F/K, where $F = \Bbbk[t]^{\oplus n}$ and $K \subset F$ is the kernel of surjection $F \twoheadrightarrow M$ sending basic vectors of F to generators of M. The result follows at once from the *elementary divisors theorem*: each submodule $K \subset F$ is free and there exist some bases $\{e_1, e_2, \dots, e_n\} \subset F$, $\{u_1, u_2, \dots, u_m\} \subset K$ and $f_1, f_2, \dots, f_m \in \Bbbk[t]$ such that $u_i = f_i \cdot e_i$ for $1 \leqslant i \leqslant m$ (moreover, f_i divides f_j for $i < j$ and the set of those elementary divisors does not depend on a choice of bases). Indeed, theorem forces $F/K = \Bbbk[t]^{\oplus(n-m)} \oplus T$, where $T = \bigoplus\limits_i (\Bbbk[t]/(f_i))$ is torsion submodule. The elementary divisors theorem also holds over any principal ideal domain (see Artin's or Van der Warden's » Algebra« textbook; special proof for either $\mathbb{Z}$ or $\Bbbk[t]$ is extremely fruitful exercise on the Gauss diagonalization and Euclid's division algorithms).

which are defined over $\mathbb{A}_1$ and over $U_\infty = \{\infty\} \cup (\mathbb{A}_1 \setminus \{0\})$. These trivialization are expressed through each other over $\mathbb{A}_1 \setminus \{0\}$ as

$$(e_1^\infty, e_2^\infty, \dots, e_r^\infty) = (e_1^0, e_2^0, \dots, e_r^0) \cdot \varphi , \tag{15-2}$$

where φ is the transition matrix whose entries are rational functions of t without zeros and poles in $\mathbb{A}_1 \setminus \{0\}$, i. e. some polynomials in t, t^{-1}. Replacing E by $E(m) \stackrel{\text{def}}{=} E \otimes \mathcal{O}(m)$, we multiply all entries of φ by t^m. We can chose m such that the first column of φ has no negative powers of t but does not vanish at $t = 0$. This means that e_1^∞ becomes nowhere vanishing global section of E over $\mathbb{P}_1$.

Exercise 15.5. Show that $\Gamma(\mathbb{P}_1, E(m)) = 0$ for $m \ll 0$.

Let us fix the minimal m such that $E(m)$ admits some nowhere vanishing global section e and replace E by $E(m)$ for this m. Thus, we will assume that $\Gamma(\mathbb{P}_1, E(d)) = 0$ for all $d < 0$.

Using induction over r, we can suppose that the factor bundle $Q = E/e \cdot \mathcal{O}$ splits as

$$Q = \mathcal{O}(d_2) \oplus \mathcal{O}(d_3) \oplus \cdots \oplus \mathcal{O}(d_r) , \quad \text{where } d_2 \leqslant d_3 \leqslant \cdots \leqslant d_r$$

By ex. 15.4, we can chose trivializations (15-2) such that $e_1^0 = e_1^\infty = e$. Then the transition rule takes a form

$$(e_1^\infty, e_2^\infty, \dots, e_r^\infty) = \left(e_1^0, e_2^0, \dots, e_r^0\right) \cdot \begin{pmatrix} 1 & f_2 & f_3 & f_4 & \dots & f_r \\ 0 & t^{d_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & t^{d_3} & 0 & \dots & 0 \\ 0 & 0 & 0 & t^{d_4} & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & t^{d_r} \end{pmatrix}$$

where $f_\nu = f_\nu(t, t^{-1})$ are some polynomials in t, t^{-1}. Moreover, by appropriate change of e_ν^∞ with $\nu \geqslant 2$ we can put all f_ν into ideal $(t) \subset \Bbbk[t]$. Indeed, it is enough to add the first column multiplied by appropriate polynomials in t^{-1} to the other columns.

As soon all $f_\nu \in (t) \subset \Bbbk[t]$, we should have all $d_\nu \leqslant 0$. Indeed, if $d_\nu > 0$ for some ν, then e_ν^∞ is extended to nowhere vanishing section of $E(-d)$ with $d = \gcd(t^{d_\nu}, f_\nu) > 0$. But this contradicts to the assumption made before. Now we can annihilate all f_ν by adding to the first row with other rows multiplied by appropriate polynomials in t (this corresponds to an invertible change of e_1^0). The resulting transition matrix becomes diagonal as required.

□

Task 1. Projective spaces.

Problem 1.1. Let S^dV^* be the space of all homogeneous degree d polynomials on n-dimensional vector space V. Find $\dim S^dV^*$.

Problem 1.2 (Veronese map). Under the previous problem conditions, let $V^* \xrightarrow{v_d} S^dV^*$ take a linear form $\psi \in V^*$ to its d-th power $\psi^d \in S^dV^*$. Does the image of v_d lie in a hyperplane or its linear span is the whole of S^dV^*?

Problem 1.3. Consider the projective closures of affine curves

a) $y = x^2$ b) $y = x^3$ c) $y^2 + (x-1)^2 = 1$ d) $y^2 = x^2(x+1)$

Write down their homogeneous equations and their affine equations in two other standard affine charts on $\mathbb{P}_2$. Try to draw all these affine curves.

Problem 1.4. Let the real Euclidian plane $\mathbb{R}^2$ be included in $\mathbb{CP}_2$ as the real part of the standard affine chart U_0. Find two points of $\mathbb{CP}_2$ such that any Euclidean circle will contain them after comlexification and projective closuring.

Problem 1.5 (Pythagorean triples). Consider $\mathbb{P}_2$ with homogeneous coordinates $(t_0 : t_1 : t_2)$. Let $\ell \subset \mathbb{P}_2$ be the line $t_2 = 0$, $Q \subset \mathbb{P}_2$ be the conic $t_0^2 + t_1^2 = t_2^2$, and $O = (1:0:1) \in Q$. For each $P = (p:q:0) \in \ell$ find coordinates of the intersection point $Q \cap (OP)$ different from O and show that the projection from O maps Q bijectively onto ℓ. Find some polynomials $a(p,q)$, $b(p,q)$, $c(p,q)$ whose values on $\mathbb{Z} \times \mathbb{Z}$ give, up to a common factor, all integer Pythagorian triples $a^2 + b^2 = c^2$ (and only such the triples).

Problem 1.6 (projecting twisted cubic). Let $\mathbb{P}_1 = \mathbb{P}(U^*)$ be the space of linear forms (up to proportionality) in two variables (t_0, t_1) and $\mathbb{P}_3 = \mathbb{P}(S^3V^*)$ be the space of cubic forms (up to proportionality) in (t_0, t_1). An image of the Veronese map $\mathbb{P}_1 \xhookrightarrow{v_3} \mathbb{P}_3$ is called *a twisted cubic* and is denoted by $C_3 \subset \mathbb{P}_3$ (comp. with Problem 1.5). Describe a projection of C_3:

a) from the point t_0^3 to the plane spanned by $3\,t_0^2t_1$, $3\,t_0t_1^2$, and t_1^3

b) from the point $3\,t_0^2t_1$ to the plane spanned by t_0^3, $3\,t_0t_1^2$, and t_1^3

c) from the point $t_0^3 + t_1^3$ to the plane spanned by t_0^3, $3\,t_0^2t_1$, and $3\,t_0t_1^2$

More precisely, write an explicit parametric representation for the projection in appropriate coordinates, then find its affine and homogeneous equation. Do that for several affine charts on the projection target plane. In each case, find degree of the the curve and try to draw it. Has it selfintersections and/or cusps?

Problem 1.7. Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$ of q elements. How many a) basises b) k-dimensional subspaces are there in V? c) How many points are there in $\mathbb{P}(V)$?

Problem 1.8*. Let $G_n^k(q)$ be a number of k-dimensional vector subspaces in n-dimensional vector space over a finite field of q elements. Compute $\lim\limits_{q \to 1} G_n^k(q)$.

Problem 1.9. Let $f : \mathbb{P}(V) \longrightarrow \mathbb{P}(V)$ be a projective linear isomorphism induced by some linear isomorphism $\widehat{f} : V \longrightarrow V$, $\dim V = n+1$. Assume that all fixed points of f are isolated. Estimate a number of them.

Task 2. Quadrics and conics.

Problem 2.1. Consider the quadratic form $q(A) = \det A$ on the space of square 2×2-matrices $\mathrm{Mat}_2(\Bbbk)$. Describe its polarization, i.e. what is the bilinear form of *two* 2×2-matrices[1] $\widetilde{q}(A, B)$ such that $\widetilde{q}(A, A) = \det(A)$?

Problem 2.2 (continuation of Problem 1.5). Under the conditions of Problem 1.5, show that any conic on $\mathbb{CP}_2$, which pass through two points you have found in Problem 1.5 and has at least 3 points inside the initial real Euclidian plane, looks there as a circle.

Problem 2.3 (Euclidean polarities). Consider a circle in the real Euclidean affine plane. How to draw[2]:

a) the polar of a given point (especially, when the point is inside the circle)

b) the pole of a given line (especially, when the line does not intersect the circle)

Describe geometrically a polarity w.r.t. an 'imaginary circle' $x^2 + y^2 = -1$.

Problem 2.4. Show that all conics passing through the points $a = (1 : 0 : 0)$, $b = (0 : 1 : 0)$, $c = (0 : 0 : 1)$, $d = (1 : 1 : 1)$ form a line in the space of all conics. Write an explicit equation[3] for these conic family and find all singular conics inside it.

Problem 2.5 (1−1 correspondence on a conic). Let $Q \subset \mathbb{P}_2$ be a smooth conic considered together with some fixed rational parameterization $\mathbb{P}_1 \xrightarrow{\sim} Q$. Show that for any bijection $Q \xrightarrow{\gamma} Q$ induced by a linear automorphism of $\mathbb{P}_1$ there exist two points $p_1, p_2 \in Q$ and a line $\ell \subset \mathbb{P}_2$ such that $x \overset{\gamma}{\longmapsto} y$ iff $\pi_\ell^{p_1} x = \pi_\ell^{p_2} y$. Were are the fixed points of this map? Is it possible, using only the ruler, to find (some) p_1, p_2, ℓ for γ given by its action on 3 points $a, b, c, \in Q$?

Problem 2.6*. Using only the ruler, draw a triangle inscribed in a given non singular conic Q and such that his sides a, b, c pass through 3 given points A, B, C. How many solutions may have this problem?

HINT. Start 'naive' drawing from any $p \in Q$ and denote by $\gamma(p)$ your return point after passing trough A, B, C. Is $p \longmapsto \gamma(p)$ a projective isomorphism of kind described in Problem 2.5?

Problem 2.7*. Formulate and solve projectively dual problem to the previous one.

Problem 2.8*. Describe a general algorithm for reducing a trigonometric equation $f(\sin(x), \cos(x)) = 0$, where f is an arbitrary quadratic polynomial in two variables, to a simple equation $\cos(x) = \alpha$, where α contains at most cubic irrationalities.

HINT. The problem is how to compute explicitly 4 intersection points of 2 quadrics $f(x, y) = 0$ and $x^2 + y^2 = 1$; but the same intersection points can be produced by any two quadrics from the same pencil. A good idea is to intersect two singular conics of this pencil.

Problem 2.9. Consider two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ and denote by $\ell_1^\times, \ell_2^\times \subset \mathbb{P}_3^\times$ two pencils of planes passing through these lines. Take any 3 non collinear points a, b, c such that no two of them are coplanar with either ℓ_1 or ℓ_2. Write $\ell_1^\times \xrightarrow{\gamma_{abc}} \ell_2^\times$ for a linear projective isomorphism that sends 3 planes passing through a, b, c in $\ell_1^\times$ to the similar planes in $\ell_2^\times$. Describe the incidence graph

$$\Gamma_{abc} \overset{\text{def}}{=} \bigcup_{\pi \in \ell_1^\times} \big(\pi \cap \gamma_{abc}(\pi)\big)$$

ruled by the intersection lines of γ_{abc}-incident planes, if: a) $\ell_1 \cap \ell_2 = \varnothing$ b) $\ell_1 \cap \ell_2 \neq \varnothing$

Problem 2.10. How many lines cross each of 4 given pairwise skew lines in: a) $\mathbb{CP}_3$ b) $\mathbb{RP}_3$ c*) $\mathbb{C}^3$ d*) $\mathbb{R}^3$? Find all possible answers and indicate those are stable under small perturbations of 4 given lines.

[1] for example, the standard Euclidean norm $\|\,(a_{ij})\,\| \overset{\text{def}}{=} \sum a_{ij}^2$ is polarized to $(A, B) = \operatorname{tr}\,(A \cdot {}^tB)$; one could expect that polarization of $\det(A)$ should look quite similarly with something else instead of ${}^tB$ ...

[2] using ruler and compasses

[3] it should be a quadratic form whose coefficients depend linearly on two homogeneous parameters

Task 3. Some multilinear algebra.

Problem 3.1. Is it true that any rank 1 matrix of size $m \times n$ can be written as a product of some $m \times 1$ and $1 \times n$ matrices?

Problem 3.2. Let $\{e_1, e_2, \dots, e_d\} \subset V$ and $\{x_1, x_2, \dots, x_d\} \subset V^*$ be dual bases. Does the tensor $\sum\limits_{\nu} x_\nu \otimes e_\nu \in V^* \otimes V$ depend on a choice of the dual bases?

Problem 3.3. Let $A \in \mathrm{Hom}(U, V) \simeq U^* \otimes V$, $B \in \mathrm{Hom}(V, W) \simeq V^* \otimes W$ be two linear maps decomposed as $A = \sum \alpha_\nu \otimes a_\nu$, $B = \sum \beta_\mu \otimes b_\mu$ with $\alpha_\nu \in U^*$, $a_\nu \in V$, $\beta_\mu \in V^*$, $b_\mu \in W$. Decompose similarly their product $B \circ A \in \mathrm{Hom}(U, W) \simeq U^* \otimes W$.

Problem 3.4. Check for any vector space V a series of canonical isomorphisms:

$$\mathrm{Hom}(V, V) \simeq V^* \otimes V \xrightarrow{\ \tau\ } (V \otimes V^*)^* \simeq \mathrm{Hom}(V, V)^*$$

where τ takes $\xi \otimes v$ to a linear form that sends $v' \otimes \xi'$ to the full contraction $\xi(v')\xi'(v)$. The resulting correlation $\mathrm{Hom}(V, V) \xrightarrow{\ \sim\ } \mathrm{Hom}(V, V)^*$ corresponds to some bilinear form $t(A, B) \overset{\text{def}}{=} \tau A(B)$ on $\mathrm{Hom}(V, V)$. Is this form symmetric? How it looks in terms of matrices? What is the corresponding quadratic form?

Problem 3.5. Let $A = (a_{ij})$ be $n \times n$ - matrix whose entries are considered as independent variables. Fix a collection of m matrix elements $a_{i_\nu j_\nu}$, where $1 \leqslant \alpha \leqslant m$. Compute $\frac{\partial^m \det A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_m j_m}}$ for:

a) $m = 1$; b) $m = 2$; c*) any m. d*) Is the Taylor expansion (15-1), written below, correct?

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \lambda^p \mu^q \cdot \sum_{\substack{I\,J: \\ \#I = \#J = p}} (-1)^{|I|+|J|} a_{IJ} b_{\widehat{I}\widehat{J}} \,. \tag{15-1}$$

Here $I = (i_1, i_2, \dots, i_p)$, $J = (j_1, j_2, \dots, j_p)$, $\widehat{I} = \{1, \dots, n\} \setminus I$, $\widehat{J} = \{1, \dots, n\} \setminus J$, (a_{IJ}) is $p \times p$-minor of A situated in I-rows and J-columns, and $\left(b_{\widehat{I}\widehat{J}}\right)$ is the complementary $q \times q$-minor of $B = (b_{ij})$.

HINT. Use the Sylvester relations relations: let A_m be $\binom{n}{m} \times \binom{n}{m}$ matrix whose entries are $m \times m$-minors of A and write $\widehat{A}_m$ for a matrix of algebraic complements to the entries of A_m; then $\det A = \binom{n}{m}^{-1} \mathrm{tr}\left(A_m \cdot {}^t\widehat{A}_m\right)$ and the rightmost sum in (15-1) equals $\mathrm{tr}\left(A_p \cdot {}^t\widehat{B}_q\right)$.

Problem 3.6. Is there a 2×4 - matrix whose 2×2 - minors are: a) $\{2, 3, 4, 5, 6, 7\}$ b) $\{3, 4, 5, 6, 7, 8\}$ (If no, explain why, if yes, give an explicit example.)

Problem 3.7. Are the following decompositions valid for any vector space V over a field of zero characteristic: a) $V^{\otimes 2} \simeq S^2 V \oplus \Lambda^2 V$ b) $V^{\otimes 3} \simeq S^3 V \oplus \Lambda^3 V$? If yes, give a proof, if no, give an explicit example of a tensor that can not be decomposed in this way.

Problem 3.8 (spinor decomposition). Let $V = \mathrm{Hom}(U_-, U_+)$, where $\dim U_\pm = 2$. Show that

$$V^{\otimes 2} = \underbrace{\left(\left(S^2 U_-^* \otimes S^2 U_+\right) \oplus \left(\Lambda^2 U_-^* \otimes \Lambda^2 U_+\right) \right)}_{S^2 V} \bigoplus \underbrace{\left(\left(S^2 U_-^* \otimes \Lambda^2 U_+\right) \oplus \left(\Lambda^2 U_-^* \otimes S^2 U_+\right) \right)}_{\Lambda^2 V} \,.$$

HINT. Write $V = U_-^* \otimes U_+$ and use the decomposition $U_\pm^{\otimes 2} = S^2 U_\pm \oplus \Lambda^2 U_\pm$.

Task 4. More quadrics and other hypersurfaces.

Problem 4.1. Let $G \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a non singular quadric given by a quadratic form g whose polarization is $\widetilde{g}$. Show that bilinear form $\Lambda^2\widetilde{g}$ on $\Lambda^2 V$, which acts on decomposable bivectors as

$$\Lambda^2\widetilde{g}(\, v_1 \wedge v_2 \,,\, w_1 \wedge w_2 \,) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1, w_1) & \widetilde{g}(v_1, w_2) \\ \widetilde{g}(v_2, w_1) & \widetilde{g}(v_2, w_2) \end{pmatrix} ,$$

is symmetric and non degenerate, and write its explicit Gram matrix in a convenient base (say, coming from an orthonormal base for g in V). Show that the intersection of the corresponding quadric $\Lambda^2 G \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ with the Plücker quadric consists of all tangent lines to $G \subset \mathbb{P}_3$.

Problem 4.2. Under the previous problem notations, let $\mathrm{Gr}(2, V)$ be the Grassmannian variety, of lines in $\mathbb{P}_3 = \mathbb{P}(V)$. Show that the Plücker embedding $\mathrm{Gr}(2, V) \hookrightarrow \mathbb{P}(\Lambda^2 V)$ sends two line families living on the Segre quadric $G \subset \mathbb{P}(V) = \mathbb{P}(\mathrm{Hom}(U_-, U_+))$ to a pair of non singular plane conics that are cut out the Plücker quadric $P \subset \mathbb{P}(\Lambda^2 V)$ by two complementary planes $\Lambda_- = \mathbb{P}\left(S^2 U_-^* \otimes \Lambda^2 U_+\right)$ and $\Lambda_+ = \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ laying in $\mathbb{P}(\Lambda^2 \mathrm{Hom}(U_-, U_+))$ via Problem 3.5. Moreover, the both conics are embedded into these planes via Veronese, that is, we have the following commutative diagram (Plücker is dotted, because it maps lines into points):

$$\begin{tikzcd}
\mathbb{P}(U_+) \arrow[rr, hook, "\text{Veronese}"] & & \mathbb{P}(S^2 U_+) \simeq \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"'] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\mathrm{Hom}(U_-, U_+) \arrow[r, dashed, "\text{Plücker}"] & P \subset \mathbb{P}\begin{pmatrix} \Lambda^2 U_-^* \otimes S^2 U_+ \\ \oplus \\ S^2 U_-^* \otimes \Lambda^2 U_+ \end{pmatrix} \\
\mathbb{P}(U_-^*) \arrow[rr, hook, "\text{Veronese}"'] & & \mathbb{P}(S^2 U_-^*) \simeq \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

Problem 4.3. Let us fix a 2-dimensional plane $\pi \subset \mathbb{P}_n$ and a pair of codimension 2 subspaces $L_1, L_2 \subset \mathbb{P}_n$ such that $p_1 = L_1 \cap \pi$ and $p_2 = L_2 \cap \pi$ are two distinct points on π. Write $\ell_1 = L_1^\times \subset \mathbb{P}_n^\times$, $\ell_2 = L_2^\times \subset \mathbb{P}_n^\times$ for two pencils of hyperplanes passing through L_1, L_2 respectively and take any $a, b, c \in \pi$ such that any 3 of 5 points p_1, p_2, a, b, c are non-collinear. Then we get a projective linear isomorphism $\gamma_{abc} \colon \ell_1 \xrightarrow{\sim} \ell_2$ defined by a, b, c like in Problem 2.5. Show that its incidence graph

$$\bigcup_{H \in \ell_1} (H \cap \gamma_{abc}(H)) \subset \mathbb{P}_n$$

is a quadric, find its rank, and describe its singular points in both possible cases:
a) $\dim(L_1 \cap L_2) = (n-3)$ b) $\dim(L_1 \cap L_2) = (n-4)$.

Problem 4.4. Let $S \subset \mathbb{P}_5 = \mathbb{P}(S^2 V^*)$ be the space of singular conics on $\mathbb{P}_2 = \mathbb{P}(V)$. Show that singular points of S correspond to double lines in $\mathbb{P}(V)$ and $\mathrm{Sing}\,(S)$ coincides with an image of the Veronese embedding $\mathbb{P}(V^*) \xhookrightarrow{v_2} \mathbb{P}_5$. For non singular $q \in S$, which corresponds to splitted conic $\ell_1 \cup \ell_2 \subset \mathbb{P}(V)$, prove that the tangent space $T_q S$, for S at q, consits of all conics passing through $\ell_1 \cap \ell_2$.

Problem 4.5. Let $S \subset \mathbb{P}_3$ be a surface ruled by all lines tangent to the twisted cubic $C_3 \subset \mathbb{P}_3$. Write down an explicit equation for S, find its degree and all singular points.

Problem 4.6. Find all lines on a singular projective cubic surface with affine equation $xyz = 1$.
Hint. Show that there are no lines in the initial affine chart

Task 5. Plane curves.

Problem 5.1 (plane cubics).

a) How many singular points may have a plane cubic curve and what could be their multiplicities?

b) Classify all reducible cubics up to a projective linear isomorphism.

c) Show that irreducible singular cubics are rational and (up to a projective linear isomorphism) are exhausted by $y^2 = x^3$ (nodal cubic) and $y^2 = x^2(x+1)$ (cuspidal cubic).

HINT. Rationality may be proved via projection from a singular point.

d) How many tangent lines come to a smooth cubic curve from a generic point on $\mathbb{P}_2$?

e) How many inflection points are there on a smooth cubic?

f*) Show that any non singular cubic may be presented in appropriate affine coordinates by equation $y^2 = x^3 + px + q$.

HINT. See: *C. H. Clemens. A scrapbook of complex curve theory.* Plenum Press. But try to simplify (or to modify) the arguments by your own geometric and/or multilinear arguments

g*) Show that 3 non-inflection tangents which are drown from an inflection point on a smooth cubic meet this cubic in 3 collinear points.

HINT. Look at the Clemens book (*loc. cit.*) but make his arguments more solid by adding your own details

Problem 5.2. Let a curve $C \subset \mathbb{A}_2$ be given by by equation $x^2y + xy^2 = x^4 + y^4$.

a) What kind of singularity has C at the origin?

b) Has the projective closure of C any other singularities (say, at the infinity) ?

c) Find a local intersection multiplicity at the origin between C and a curve with a simple cusp whose cuspidal tangent is $x = y$.

Problem 5.3. For plane curves a) $(x_0 + x_1 + x_2)^3 = 27\, x_0x_1x_2$ b) $(x^2 - y + 1)^2 = y^2(x^2 + 1)$ find all[1] singular points, compute their multiplicities, look how many branches come to each singularity and what are their geometric tangents.

HINT. To analyze local geometry, blow up the singularity, i. e. take affine coordinates (x, y) centered at the singularity and substitute $x = \alpha\, t$, $y = \beta\, t$ in the equation of curve; then the geometric tangent lines have slopes $(\alpha : \beta)$ for which a multiplicity of the zero root $t = 0$ jumps.

Problem 5.4. Using the Plücker relations, list all complex plane quartics with the simplest singularities (i. e. ordinary double nodes and cusps only) w. r. t. how many cusps, nodes, double tangents and inflexion points may they have. Which of them have to be reducible?

Problem 5.5. Describe all complex plane projective quintics that have singularities of multiplicity 4 at two given distinct points $a, b \in \mathbb{P}_2$.

HINT. They have to contain a (multiple) line (a, b) and form 3-dimensional projective space.

Problem 5.6. For a curve $C \subset \mathbb{P}_2$ of degree d curve let us fix some point $q \notin C$ that does not lie either on an inflection tangent or on a geometric tangent through a singular point of C. Write $C_q^{(d-1)}$ for $(d-1)$-th degree polar of q w. r. t. C. Compute a local intersection index $(C, C_q^{(d-1)})_p$ at a point $p \in C$ when

a) p is smooth; b) p is an ordinary cusp; c) p is an ordinary m-typle node $m(m-1)$.

HINT. In (a), (b) p is smooth on $C_q^{(d-1)}$ as well and $T_pC_q^{(d-1)} \neq T_pC$ in (a) but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangent in (b). In (c) p is an $(m-1)$-typle point on $C_q^{(d-1)}$ but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$ and hence intersects it with multiplicity $(m-1)$.

Problem 5.7. Show that smooth plane quartic curve either has a tangent line intersecting the curve just ones with multiplicity 4 or has 28 bitangent lines (touching the curve in two distinct points).

[1] including possible singularities at the infinity in (b)

Task 6. Polynomial ideals.

Problem 6.1. Give an example of proper non-principal ideal in a) $\mathbb{C}[x,y]$ b) $\mathbb{Z}[x]$.

Problem 6.2. Let a polynomial f vanish along a hypersurface given in $\mathbb{C}^n$ by a polynomial equation $g=0$. Prove that each irreducible factor of g divides f.

Problem 6.3. Prove that any algebraic set in $\mathbb{C}^2$ is a finite union of points and curves (recall that a curve is a zero set of one polynomial).

Problem 6.4. Let $J=(xy,yz,zx)\subset\mathbb{C}[x,y,z]$. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$. Is it possible to define the same variety by 2 polynomial equations?

Problem 6.5. Find $f\in I(V(J))\setminus J$ for $J=(x^2+y^2-1,y-1)\subset\mathbb{C}[x,y]$.

Problem 6.6. Describe $V(J)\subset\mathbb{A}^3$ and $I(V(J))\subset\mathbb{C}[x,y,z]$ for:
a) $J=(xy,(x-y)z)$ b) $J=(xy+yz+zx,x^2+y^2+z^2)$

Problem 6.7. Which of the following three facts about ideals in $\Bbbk[x_1,x_2,\dots,x_n]$ (where $\Bbbk$ is an arbitrary field) are true? (Prove the true ones and give counter-examples for the other.)
a) $\sqrt{IJ}=\sqrt{I\cap J}$
b) $\sqrt{IJ}=\sqrt{I}\sqrt{J}$
c) $(I=\sqrt{I}\ \&\ J=\sqrt{J})\quad\Rightarrow\quad IJ=\sqrt{IJ}$

Problem 6.8. Let $B\supset A$ be an extension of commutative rings such that B is finitely generated as A-module. Prove that $\mathfrak{m}B\neq B$ for any maximal ideal $\mathfrak{m}\subset A$.

Problem 6.9. Which of the following three rings are Noetherian?
a) $\left\{ f(z)=\dfrac{p(z)}{q(z)}\in\mathbb{C}(z)\ \middle|\ q(z)\neq 0 \text{ for } |z|\leqslant 1\right\}$;
b) power series $f(z)\in\mathbb{C}[[z]]$ converging everywhere on $\mathbb{C}$;
c) $\left\{ f(x,y)\in\mathbb{C}[x,y]\ \middle|\ \dfrac{\partial^{i+j}f}{\partial x^i\partial y^j}=0\quad\forall\, 0\leqslant i+j\leqslant n\right\}$, where $n\in\mathbb{N}$ is fixed.

Problem 6.10*. Show that any finitely generated[1] field is finite as a set.

Problem 6.11*. Show that an ideal $I(C_3)$, which is generated by all homogeneous $f\in\mathbb{C}[x_0,x_1,x_2,x_3]$ vanishing along the twisted cubic $C_3\subset\mathbb{P}_3$
a) is generated by 3 quadratic polynomials b) can't be generated by 2 polynomials

[1] in absolute sense, i. e. as $\mathbb{Z}$-algebra w. r. t. the action $m\cdot a\overset{\text{def}}{=}\underbrace{a+a+\,\cdots\,+a}_{m \text{ times}}$

Task 7. Algebraic manifolds.

Problem 7.1 (Zariski topology). Let $X = \operatorname{Spec}_{\mathrm{m}} A$ be affine algebraic set. Check that the sets

$$V(I) = \{\, x \in X \mid f(x) = 0 \ \ \forall\, f \in I \,\}$$

produced by all ideals $I \subset A$ satisfy the closed sets axioms of the topology.

Problem 7.2. Prove that any open covering of affine algebraic variety contains a finite sub-covering.

Problem 7.3. Give an example if affine algebraic set X and open $U \subset X$ such that $\mathscr{O}_X(U)$ is not finitely generated as $\Bbbk$-algebra.

Problem 7.4. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets.

a) Show that $X \times Y$ is affine algebraic subset in $\mathbb{A}^{n+m}$.

b) Give $X \times Y \subset \mathbb{A}^{n+m}$ by explicit equations (assuming that the equations for X, Y are known).

c) Show that $X \times Y$ is irreducible as soon both X, Y are.

Problem 7.5. Prove that the maximal spectrum of a finite dimensional[1] $\Bbbk$-algebra is a finite set and deduce from this that any finite morphism has only finite (or empty) fibers.

Problem 7.6. Give an example of regular morphism of affine algebraic sets $X \xrightarrow{\varphi} Y$ such that all fibers of φ are finite (or empty) but φ is not a finite morphism.

Problem 7.7. Prove that a projection of affine hypersurface $V(f) \subset \mathbb{A}^n$ from any point $p \notin V(f)$ onto any hyperplane $H \not\ni p$ is dominant.

Problem 7.8 (Noether's normalization). Show that any affine hypersurface $V(f) \subset \mathbb{A}^n$ admits a finite surjection onto some hyperplane $\mathbb{A}^{n-1} \subset \mathbb{A}^n$.

Problem 7.9. Prove that $\dim(X \times Y) = \dim X + \dim Y$

Problem 7.10. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of algebraic manifolds. Show that isolated[2] points of fibers $\varphi^{-1}(y)$ draw an open subset of X when y runs through Y.

Hint. Use Chevalley's theorem on semi-continuity (lecture 13).

Problem 7.11. Show that an image of a regular dominant morphism contains an open dense subset.

Problem 7.12* (Chevalley's constructivity theorem). Prove that an image of any regular morphism of algebraic varieties is *constructive*, i. e. can be constructed from a finite number of open and closed subsets by a finite number of unions, intersections, and taking complements.

Problem 7.13 (quadratic transformation). Show that the prescription $(t_0 : t_1 : t_2) \longmapsto (t_0^{-1} : t_1^{-1} : t_2^{-1})$ is extended to a rational map $\mathbb{P}_2 \overset{q}{\dashrightarrow} \mathbb{P}_2$ defined everywhere except for 3 points; find these points; clarify how does q act on a triangle (triple of lines) with the vertexes at these 3 points; find $\operatorname{im} q$.

Problem 7.14 (graph of rational map). Let $X \overset{\psi}{\dashrightarrow} Y$ be a rational map defined on open dense $U \subset X$. By the definition, its *graph* $\Gamma_\psi \subset X \times Y$ is the Zariski closure of $\{(x, \psi(x)) \in X \times Y \mid x \in U\}$.

a) Show that a graph of the natural rational map $\mathbb{A}^{n+1} \dashrightarrow \mathbb{P}_n$, which sends $P \in \mathbb{A}^{n+1}$ to $(OP) \in \mathbb{P}_n$, is isomorphic to the blow up of the origin.

b*) Try to describe a graph of the quadratic transformation from Problem 7.5, in particular, describe the fibers of its projections onto the both, source and target, $\mathbb{P}_2$'s.

[1] as a vector space over $\Bbbk$

[2] a point $p \in M$ is called *isolated* point of a subset $M \subset X$ in a topological space X, if it has an open neighborhood $U \ni p$ such that $U \cap M = \{p\}$

Task 8. 27 lines.

Problem 8.1 (Schläflische Doppelsechs). The 'double six line configuration' is constructed as follows. Let

$$[0],\,[1],\,\dots\,,\,[5]\subset\mathbb{P}_3$$

be six lines such that [1], ... , [5] are mutually skew, [0] intersects all of them, and each of [1], ... , [5] does not either touch or lay on the quadric drown through any 3 other. Show that: a) $\forall\ i=1,\dots,5\ \exists$ unique line $[i']\neq[0]$ such that $[i']\cap[j]\neq\varnothing\ \forall\ j\neq i$;

b) $[i']\cap[i]=[i']\cap[j']=\varnothing$ for all $i=1,\dots,5$ and for all $j\neq i$;

c) each of $[1'],\dots,[5']$ does not either touch or lay on the quadric drown through any 3 other;

d) there exists a unique line $[0']$ that intersects each of $[1'],\dots,[5']$

Hint. Let $[0'_1]\neq[1]$ and $[0'_2]\neq[2]$ be the lines, which intersect all $[1'],\dots,[5']$ except for $[1']$ and $[2']$ respectively; show that they have the same intersection points p_3, p_4, p_5 with $[3']$, $[4']$, $[5']$, which may be recovered geometrically using only the lines [3], [4], [5], $[3']$, $[4']$, $[5']$, and [0].

Problem 8.2. Show that each double six line configuration lies on a smooth cubic surface and explain how to find the other 15 lines laying on it.

Problem 8.3. Can a smooth cubic surface $S\subset\mathbb{P}_3$ have a plane section that splits into a smooth conic and its tangent line?

Problem 8.4 (projecting a smooth cubic). Let $S\subset\mathbb{P}_3$ be a smooth cubic surface, $p\in S$ be outside the lines laying on S, $\pi\not\ni p$ be any plane, and $Q=\{q\in\pi\mid(pq)\text{ touches }S\text{ outside }p\}$ be the apparent contour of S visible from p and projected from p onto π. Show that:

a) each plane section passing through p and any line $\ell\subset S$ contains precisely 2 distinct tangent lines coming from p onto S;

Hint. Look at the (smooth!) residue conic $(S\cap(p\ell))\setminus\ell$.

b) $Q\subset\pi$ is a smooth quartic;

Hint. Look at the discriminant of $S|_{(pq)}\setminus\{p\}$.

c) Q has precisely 28 distinct double tangents, which are exhausted by $T_pS\cap\pi$ and projections (from p onto π) of lines laying on S;

Hint. Use the Plücker relation to compute the number of bitangents.

d) deduce from the previous assertions a new proof of the existence of precisely 27 lines on a smooth cubic surface are projections of 27 lines laying on S.

Problem 8.5*. Show that any smooth cubic $S\subset\mathbb{P}_3$ can be given in appropriate coordinate system by equation $\varphi_1\varphi_2\varphi_3+\psi_1\psi_2\psi_3=0$, where all φ_i, ψ_j are linear homogeneous forms.

Hint. Use a line $\ell\subset S$ and 5 planes passing through it and intersecting S in a triple of distinct lines

Problem 8.6. Let $\ell_1,\ell_2\subset S$ be two skew lines on a smooth cubic surface $S\subset\mathbb{P}_3$. Show that the prescription:

$$p\longmapsto(\ell\cap\ell_1,\ell\cap\ell_2)$$

where $p\in S\setminus(\ell_1\cup\ell_2)$ and ℓ is a unique line through p meeting the both lines ℓ_i, can be extended to a regular morphism $S\xrightarrow{\varrho}\mathbb{P}_1\times\mathbb{P}_1=\ell_1\times\ell_2$. Show also that:

a) ϱ contracts 5 lines on S to some points on $\mathbb{P}_1\times\mathbb{P}_1$;

b) ϱ is rational isomorphism[1], i.e. there is a rational map $U\xrightarrow{\varrho^{-1}}S$ defined on some open dense $U\subset\mathbb{P}_1\times\mathbb{P}_1$ such that $\varrho\circ\varrho^{-1}=\mathrm{Id}_U$ and $\varrho^{-1}\circ\varrho=\mathrm{Id}_W$ for some open dense $W\subset S$.

Problem 8.7. Let $p\in S$ be a singular point of a (singular) cubic surface in $\mathbb{P}_3$. Show that there is at least one (but in general 6) lines laying on S and passing through p.

[1]this means, in particular, that S is rational, i.e. admits a rational parameterization

Test 1 (elementary geometry).

Problem 1.1. Find a condition on 5 lines in $\mathbb{P}_2$ necessary and sufficient for the existence of a unique non-singular conic touching all these lines.

Problem 1.2. Consider the complex plane quartic[1]

$$(x_0^2 + x_1^2)^2 + 3\,x_0^2 x_1 x_2 + x_1^3 x_2 = 0 \qquad (*)$$

a) Find all its singular points over $\mathbb{C}$.

b) Describe a local structure of each singularity (i. e. geometrical tangents and their intersection multiplicities with the curve).

c) Find a rational parameterization for C.

HINT. Use a projection from a singular point onto a line.

Problem 1.3. Show that any irreducible plain quartic with a singularity of multiplicity 3 is rational.

Problem 1.4. Consider projective plane $\mathbb{P}_2$ with homogeneous coordinates $(t_0 : t_1 : t_2)$.

a) Show that all plane quintics[2] that have an ordinary cusp (of multiplicity 2) at $(0 : 0 : 1)$ with the cuspidal tangent $t_1 = 0$ form a projective subspace in the space of all plane quintics.

b) Find the dimension of this subspace.

c) Compute local intersection multiplicity between such a quintic and quartic $(*)$ at $(0 : 0 : 1)$.

Problem 1.5. Prove that a space of homogeneous degree d polynomials (in several variables) over a field of zero characteristic is linearly generated by pure d-th powers of linear forms[3].

Problem 1.6. Let A be a finitely generated k-algebra. Show that if A is finite dimensional as a vector space over k, then $\mathrm{Spec}_{\mathrm{m}}\, A$ is a finite set.

[1] i. e. a plain curve of degree 4

[2] i. e. plane curves of degree 5

[3] this implies that any linear assertion about polynomials, e. g. the Taylor formula, is true as soon it holds for d-th powers of all linear forms

Test 2 (advanced geometry).

Problem 2.1. Prove that any hypersurface in $\mathbb{A}_n$ admits a finite surjective morphism onto $\mathbb{A}_{n-1}$.

HINT. Use appropriate projection.

Problem 2.2. Write $\mathbb{P}_N = \mathbb{P}(S^4V^*)$ for the space of quartic hypersurfaces in $\mathbb{P}_3 = \mathbb{P}(V)$ (where $\dim V = 4$). Show that all quartics containing a line form a hypersurface[1] in $\mathbb{P}_N$.

Problem 2.3. Show that any nowhere vanishing regular section of the trivial rank r vector bundle over $\mathbb{A}_1$ can included in some system of r regular sections that form a base in each fiber[2].

Problem 2.4. Consider the standard covering of the Grassmannian $\mathrm{Gr}(m,n)$, of m-dimensional subspaces in k^n, by affine charts U_I consisting of $W \subset k^n$ which are isomorphically projected onto m-dimensional subspace spanned by i_1-th, i_2-th, …, i_m-th basic vectors of k^n along all the other $(n-m)$ basic vectors[3]. Let us present a point $W \in U_I$ by $n \times m$ - matrix $M_I(W)$, whose columns are the coordinates of vectors forming a unique base of W such that $k \times k$ - submatrix situated in i_1-th, i_2-th, …, i_m-th rows is the identity matrix. We consider other $(n-m)\cdot m$ matrix elements (staying outside I-rows) as affine coordinates of W in the chart U_I. Let $S \twoheadrightarrow \mathrm{Gr}(m,n)$ be the tautological vector subbundle of $k^n \times \mathrm{Gr}(m,n)$ whose fiber over a point $W \in \mathrm{Gr}(m,n)$ is the subspace $W \subset k^n$.

a) Construct some trivializing basic sections $s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}$ for S over each U_I and describe corresponding transition matrices $\Phi_{IJ} = \Phi_{IJ}(W)$, which satisfy

$$(s_1^{(I)}, s_2^{(I)}, \dots, s_m^{(I)}) \cdot \Phi_{IJ} = (s_1^{(J)}, s_2^{(J)}, \dots, s_m^{(J)})$$

everywhere in $U_I \cap U_J \subset U_I$.

HINT. Write M_{IJ} for the $m \times m$ - submatrix of M_I situated in J-rows; then M_J is easily expressed through M_I and M_{IJ}.

b) Do the same for the line bundle $D = \Lambda^m S$ and for each its tensor power $D^{\otimes d}$.

c) Prove that any line bundle L over $\mathrm{Gr}(m,n)$ is isomorphic to some $D^{\otimes d}$.

HINT. Write D_{IJ} for determinant $\det M_{IJ}$ of the matrix introduced in the previous hint. The transition function φ_{IJ}, of L, is a rational function in matrix elements of M_I regular and non-vanishing everywhere on U_I except for the zero set of D_{IJ}.

Problem 2.5*. How many triple intersection points[4] have 27 lines on a smooth cubic surface?

[1] in other words, there is a polynomial Φ in the coefficients of variable quartic form $F(x_0 : x_1 : x_2 : x_3)$ such that $\Phi(F) = 0$ iff the quartic $F = 0$ contains a line

[2] more honorary (and not obligatory, certainly!) problem is to do the same over $\mathbb{A}_n$

[3] as usually, $I = (i_1, i_2, \dots, i_m)$ runs through all increasing collections of m elements of $\{1, 2, \dots, n\}$

[4] that is, the points where some 3 out of 27 lines are intersecting simultaneously

Actual middle term test, April 04, 2006.

Problem 1. Let U, V be 2-dimensional vector spaces and

$$Q \simeq \mathbb{P}(U^*) \times \mathbb{P}(V) \subset \mathbb{P}(U^* \otimes V)$$

be the Segre quadric formed by rank 1 linear operators $U \xrightarrow{\xi\otimes v} V$ considered up to proportionality. Show that the tangent plane $T_{\xi\otimes v}Q$ to Q at a point $\xi \otimes v \in Q$ is formed by all linear operators $U \longrightarrow V$ that send 1-dimensional subspace $\mathrm{Ann}\,(\xi) = \{u \in U \mid \xi(u) = 0\}$ into 1-dimensional subspace spanned by v.

Problem 2. Let $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$ be the space of all singular conics on $\mathbb{P}_2 = \mathbb{P}(V)$.

a) Show that the set of its singular points $\mathrm{Sing}\,(S) \subset S$ coincides with the image of Veronese embedding $\mathbb{P}(V^*) \xhookrightarrow{\psi\mapsto\psi^2} \mathbb{P}(S^2V^*)$ (i.e. with the set of all double lines in $\mathbb{P}_2$).

b) For any non-singular point $q = \{\ell_1 \cup \ell_2\} \in S$ show that the tangent space T_qS to S at q in $\mathbb{P}_5$ is formed by all conics passing through $\ell_1 \cap \ell_2$ in $\mathbb{P}_2$.

Problem 3. Let two plane curves of the same degree d have d^2 distinct intersection points. Show that if some dm of these intersection points lay on a curve of degree $m < d$, then the rest $d\,(d-m)$ points have to lay on a curve of degree $(d-m)$.

Hint. This generalizes Pascal's theorem obtained as $d = 3$, $m = 2$. Use a pencil of curves spanned by two given curves and the properties of pencils of plane curves.

Problem 4. Find the center[1] of the grassmannian algebra in m variables over a field of char $\neq 2$.

Problem 5. Is there a 2×4 matrix whose (non ordered) set of 2×2 - minors is

a) $\{\,2, 3, 4, 5, 6, 7\,\}$

b) $\{\,3, 4, 5, 6, 7, 8\,\}$

If such a matrix exists, write down some explicit example; if not, explain why.

Hint. Use the Plücker quadratic equation for $\mathrm{Gr}(2, 4) \subset \mathbb{P}_5$ and some congruence reasons (instead of direct fingering 720 possible permutations).

Problem 6. Show that any finite dimensional (as a vector space) algebra over an arbitrary field has only a finite set of prime[2] ideals and all these ideals are maximal.

Hint. Use properties of integer ring extensions when one of two rings is a field.

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[1] i. e. all elements that commute with each element of the algebra

[2] recall that an ideal $\mathfrak{p} \subset A$ is called *prime* if $A/\mathfrak{p}$ has no zero divisors

Actual final written exam, May 22, 2006.

Notes on marks. Some problems are subdivided into several questions. Complete answer on each question gives you 5 points. Problems and questions can be solved in any order. Total sum $\geqslant 35$ points is sufficient for getting the maximal examination mark «A».

Problem 1 (10 points). Let A and B be two matrices with m rows and $n \geqslant m$ columns. Prove that $\det(A \cdot B^t) = \sum_I \det A_I \det B_I$, where the sum is running over all increasing sequences $I = (i_1, i_2, \dots, i_m) \subset \{1, 2, \dots, n\}$ and A_I, B_I mean $m \times m$-submatrices formed by I-columns.

Problem 2. Let $\mathbb{P}_N = \mathbb{P}(S^2V^*)$ be the space of quadrics on $\mathbb{P}_n = \mathbb{P}(V)$ and $X \subset \mathbb{P}_N$ be the set of all singular quadrics. Show that

a) (5 points) X is an algebraic variety and $q \in X$ is smooth iff the corresponding singular quadric $Q \subset \mathbb{P}_n$ has just one singular point;

b) (5 points) for any smooth $q \in X$ the tangent space $T_qX \subset \mathbb{P}_N$ consists of all quadrics passing through the singularity of $Q \subset \mathbb{P}_n$.

Problem 3. Show that there exists a unique homogeneous polynomial P on the space of homogeneous forms of degree 4 in 4 variables such that P vanishes at f iff the surface $f = 0$ in $\mathbb{P}_3$ contains a line. To this aim:

a) (5 points) Show that all pairs $\ell \subset S$, where $\ell \subset \mathbb{P}_3$ is a line, $S \subset \mathbb{P}_3$ is a quartic surface, form a projective variety $\Gamma \subset \mathbb{P}(\Lambda^2\mathbb{C}^4) \times \mathbb{P}(S^4(\mathbb{C}^4)^*)$.

b) (5 points) Show that Γ is irreducible and find its dimension.

c) (5 points) Show that an image of projection of Γ on $\mathbb{P}(S^4(\mathbb{C}^4)^*)$ is an irreducible hypersurface.

Problem 4. Fix 6 points $\{p_1, p_2, \dots, p_6\} \subset \mathbb{P}_2 = \mathbb{P}(V)$ such that any 3 are not collinear and all 6 do not lay on the same conic. Let $W = \{F \in S^3V^* \mid F(p_i) = 0 \text{ for each } i = 1, 2, \dots, 6\}$ be the space of cubic forms on V that vanish at these 6 points. A map

$$\mathbb{P}_2 \setminus \{p_1, p_2, \dots, p_6\} \xrightarrow{\psi} \mathbb{P}(W^*)$$

takes $p \notin \{p_1, p_2, \dots, p_6\}$ to a linear form $\mathrm{ev}_p : F \longmapsto F(p)$ on W (when p is multiplied by λ this form is multiplied by λ^3, so the map between the projectivizations is well defined). Geometrically, $\mathbb{P}(W)$ is the space of cubic curves passing through $\{p_1, p_2, \dots, p_6\}$ and ψ sends p to a hyperplane $H_p \subset \mathbb{P}(W)$ formed by all such cubics passing also through p. Show that:

a) (5 points) $\dim W = 4$;

b) (5 points) $S = \overline{\psi\left(\mathbb{P}_2 \setminus \{p_1, p_2, \dots, p_6\}\right)} \subset \mathbb{P}_3 = \mathbb{P}(W^*)$ is a cubic surface;

c) (5 points) find 27 lines in $\mathbb{P}(W)$ (i. e. 27 pencils of cubics passing through $\{p_1, p_2, \dots, p_6\}$) whose dual lines in $\mathbb{P}(W^*)$ lay on S.

Problem 5 (5 points). Let a Noetherian ring A have a unique proper maximal ideal $0 \neq \mathfrak{m} \subset A$. Show that $\mathfrak{m}M \neq M$ for any non zero finitely generated A-module M.

Actual final written exam, May 20, 2008.

Notes on marks. The problems are subdivided into several questions. Complete answer on each question gives you 5 points. Problems and questions can be solved in any order. Total sum $\geqslant 35$ points is sufficient for getting the maximal examination mark «A».

Problem 1. Let us fix $(n+1)$ degrees $d_0, d_1, \dots, d_n$ and write $\mathbb{P}_{N_i} = \mathbb{P}S^{d_i}V^*$ for the space of degree d_i hypersurfaces in $\mathbb{P}_n = \mathbb{P}(V)$.

a) (5 points) Show that $\Gamma = \{(S_0, S_1, \dots, S_n, p) \in \mathbb{P}_{N_0} \times \cdots \mathbb{P}_{N_n} \times \mathbb{P}_n \mid p \in \bigcap\limits_{\nu=0}^{n} S_\nu\}$ is an irreducible projective variety.

b) (5 points) Find $\dim \Gamma$.

c) (5 points) Show that there exists a polynomial R in the coefficients of homogeneous forms $F_0, F_1, \dots, F_n$ of degrees $d_0, d_1, \dots, d_n$ in variables $(x_0, x_1, \dots, x_n)$ such that $R = 0$ iff the system of equations $F_\nu(x_0, x_1, \dots, x_n) = 0$ $(0 \leqslant \nu \leqslant n)$ has a non zero solution. How does R look like for a system of linear forms?

Problem 2. Write M for the projective space of $m \times n$ matrices considered up to proportionality. Use appropriate incidence variety $\{(L, F) \mid L \subset \ker F\}$ (where L is a subspace and F is a matrix)

a) (5 points) to show that the matrices of rank $\leqslant k$ form an irreducible projective subvariety $M_k \subset M$,

b) (5 points) to find $\dim M_k$.

Problem 3. Use the claim that an algebra A equipped with an action of a finite group G is integer over the subalgebra of G-invariants $A^G \subset A$ to solve the following problems:

a) (5 points) Let a finite group G act on an affine algebraic variety X by regular automorphisms. Construct an affine algebraic variety X/G and a finite regular surjection $X \twoheadrightarrow X/G$ whose fibers are exactly G-orbits.

b) (5 points) Show that X/G is universal in the following sense: for any regular morphism of affine algebraic varieties $X \xrightarrow{\varphi} Y$ such that $\varphi(gx) = \varphi(x)$ for all $g \in G$ and all $x \in X$ there exists a unique regular morphism $G/X \xrightarrow{\psi} Y$ such that $\psi\pi = \varphi$.

c) (5 points) Let the symmetric group $\mathfrak{S}_n$ act on the affine space $\mathbb{A}_n$ by the permutations of coordinates. Describe $\mathbb{A}_n/\mathfrak{S}_n$.

Problem 4. Let $P = \mathrm{Gr}(2,4) \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ be the grassmannian of lines in $\mathbb{P}_3 = \mathbb{P}(V)$. Show that

a) (5 points) P does not contain 3-dimensional projective subspaces;

b) (5 points) 2-dimensional planes on P are exhausted by two families parameterized by $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ respectively: a plane of the first family $\Pi_p \subset P$, $p \in \mathbb{P}(V)$, consists of all lines passing through the point p; a plane of the second family $\Pi_\pi \subset P$, $\pi \in \mathbb{P}(V^*)$, consists of all lines lying inside the plane $\pi \subset \mathbb{P}(V)$; moreover, any two planes of the same family are intersecting in one point and any two planes from divers families either have empty intersection or are intersected along some line lying on P;

c) (5 points) for any line $L \subset P$ there exist a unique pair $(p, \pi) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$ such that $L = \Pi_p \cap \Pi_\pi$.