An Analytical Expression for the Distribution of the Sum of Random Variables with a Mixed Uniform Density and Mass Function

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Abstract The distribution of the sum of independent random variables plays an important role in many problems of applied mathematics. In this paper we concentrate on the case when random variables have a continuous distribution with a discontinuity (or a probability mass) at a certain point *r*. Such a distribution arises naturally in actuarial mathematics when a responsibility or a retention limit is applied to every claim payment. An analytical expression for the distribution of the sum of i.i.d. random variables, which have a uniform distribution with a discontinuity, is reported.

1 Introduction

There are a number of problems in different fields of applied mathematics where it is required to calculate the distribution of the sum of independent random variables. This distribution for the case of uniform variables appears in such problems as handling data drawn from measurements characterized by different levels of precision, change point analysis, aggregating scaled values with differing numbers of significant figures (Buonocore et al. [2009]). An analytical expression for the distribution of non-identically distributed uniform variables is first found in Olds ([1952]). A number of subsequent works are devoted to this distribution and different proofs of its formula: Bradley and Gupta ([2002]), Sadooghi-Alvandi et al. ([2009]), Potuschak and Muller ([2009]), Buonocore et al. ([2009]).

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The solution for a simpler case of independent identically distributed uniform variables was obtained by Lagrange in the theory of geometric probabilities (Feller [1957]). This distribution is also known as Irwin-Hall distribution for two different proofs of its formula given in Irwin ([1927]) and Hall ([1927]). In the current paper we also consider the case of independent identically distributed random variables, which have a uniform distribution, but with a discontinuity (or a probability mass) at a certain point r. Such a distribution arises naturally in actuarial science, where r plays a role of a responsibility or a retention limit applied to every claim payment (Bowers et al. [1986]; Kremer [1999]). The probability density of the sum of n payments is the n-fold convolution of the mixed density and mass function. The analytical solution for a mixed exponential density and mass function is derived in Haehling von Lanzenauer and Lundberg ([1974]) by means of Laplace transform. In this paper we use an inductive procedure to get an analytical formula for the case of a mixed uniform density and mass function.

2 Uniform Distribution with Discontinuity

Let us consider a mixed uniform distribution at $[0,1]$ with a probability mass at point r (Fig. 1).

$$F(x)=\begin{cases}0, & x<0\\ x, & 0\le x\le r\\ 1, & x>r\end{cases} \quad .$$

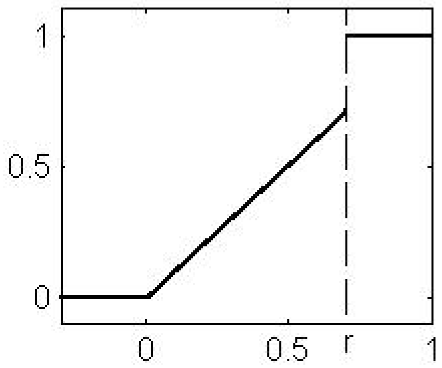

Fig. 1: Mixed uniform distribution

The distribution function of the sum $S_n = X_1 + X_2 + ... + X_n$ is denoted as $F_n(x)$:

$$X_i \sim F(x), \quad S_n \sim F_n(x) \quad .$$

The goal is to find an analytical formula for $F_n(x)$. Note that for the case r = 1 there is no discontinuity and the formula is well known (Feller [1957]):

$$F_n(x) = \frac{1}{n!}\sum_{i=0}^{k-1}(-1)^i C_n^i (x-i)^n, \quad x \in [k-1,k], \quad k = 1,2,...,n.$$

2.1 Recurrent Formula

We denote the probability of event E as $\mathrm{P}(E)$. The sums of $n+1$ and n variables X_i and their distributions are connected by the following relations:

$$\begin{aligned} S_{n+1} = S_n + X_{n+1} \sim F_{n+1}(x) = \mathrm{P}(S_n < x - X_{n+1}) \ , \\ S_n \sim F_n(s) = \mathrm{P}(S_n < s) \ , \\ X_{n+1} \sim F(t) = \mathrm{P}(X_{n+1} < t) \ . \end{aligned}$$

Since $F(x)$ has a discontinuity at point r it is necessary to find out how it is reflected on $F_n(x)$. The probability density of X_i is equal to:

$$f(t) = \begin{cases} 1 + (1-r)\,\delta(t-r), & 0 < t \le r \\ 0, & t \le 0, t > r \end{cases} \quad ,$$

where $\delta(t-r)$ is the Dirac delta function. As soon as X_i can take values only from [0, r] segment then the sum $S_n = X_1 + ... + X_n$ belongs to[0, nr] segment:

$$\begin{aligned} \mathrm{P}(S_n < 0) = \mathrm{P}(S_n > nr) = 0 \ , \\ \mathrm{P}(X_i = r) = 1 - r \ , \\ \mathrm{P}(S_n = nr) = (1-r)^n \ , \\ \mathrm{P}(S_n < nr) = 1 - (1-r)^n \ . \end{aligned}$$

As a result we have that function $F_n(x)$ has a jump of $(1-r)^n$ height at point $x = nr$:

$$F_n(x) = \begin{cases} 0, & x \le 0 \\ F_n(x), & 0 < x < nr \\ 1 - (1-r)^n, & x = nr \\ 1, & x > nr \end{cases}$$

Here and after we will consider function $F_n(x)$ only on $[0, nr]$ segment.

Lemma 1. *The following recurrent formula is true for the sum distribution function* $F_n(x)$

$$F_{n+1}(x)=\begin{cases}\int\limits_0^x F_n(t)dt, \quad 0\le x\le r\\ \int\limits_{x-r}^{x} F_n(t)dt+(1-r)F_n(x-r), \quad r\le x\le nr\\ \int\limits_{x-r}^{nr} F_n(t)dt+(x-nr)+(1-r)F_n(x-r), \quad nr\le x\le (n+1)r\end{cases}$$

Proof. To find distribution function $F_{n+1}(x)=\mathrm{P}(S_n<x-X_{n+1})$ it is necessary to calculate the integral of the joint probability density of S_n and X_{n+1} over region $D: S_n<x-X_{n+1}$ (see Fig. 2).

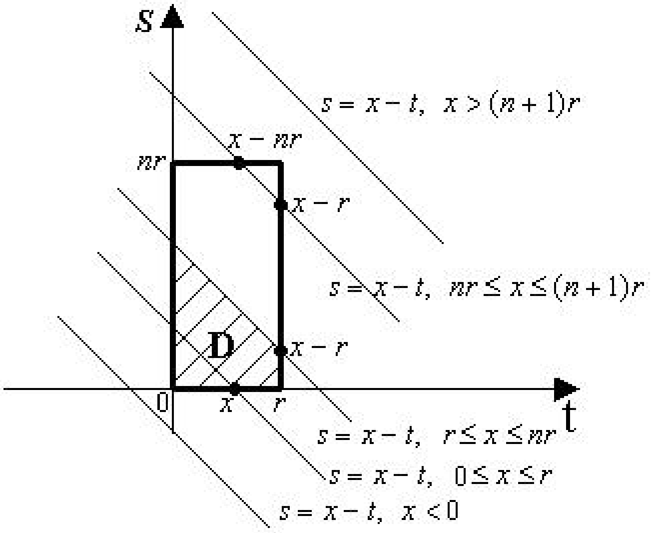

Fig. 2: Integration region for the case $x\le nr$

We calculate $F_{n+1}(x)$ on $[0,\ (n+1)r]$ segment taking into account the special values $s=nr$ and $t=r$:

$$\begin{aligned}\mathrm{P}\left(S_n<x-X_{n+1}\right)=\mathrm{P}\left((S_n<x-X_{n+1})\&(S_n\ne nr)\&(X_{n+1}\ne r)\right)\\ +\,\mathrm{P}\left((X_{n+1}=r)\&(S_n<xr)\&(S_n\ne nr)\right)\\ +\,\mathrm{P}\left((S_n=nr)\&(X_{n+1}<x-nr)\&(X_{n+1}\ne r)\right)\ ,\end{aligned}$$

$$F_{n+1}(x)=\iint\limits_D f_n(s)f(t)dsdt+(1-r)F_n(x-r)+(1-r)^nF(x-nr)\ .$$

Note that $F(x-nr)=0$ for $x<nr$ and thus:

$$F_{n+1}(x) = \begin{cases} \iint\limits_D f_n(s)f(t)dsdt + (1-r)F_n(x-r) + (1-r)^n(x-nr), \\ \quad nr \le x \le (n+1)r \\ \\ \iint\limits_D f_n(s)f(t)dsdt + (1-r)F_n(x-r), \quad 0 \le x \le nr \end{cases}$$

To get the integral over region D we'll consider 3 cases:
1) $r \le x \le nr$ (see Fig. 2):

$$\iint\limits_D f_n(s)f(t)dsdt = \int\limits_0^r dt \int\limits_0^{x-t} f_n(s)ds = \int\limits_0^r F_n(x-t)dt = \int\limits_{x-r}^{x} F_n(t)dt \quad .$$

2) $0 \le x \le r$ (see Fig. 2):

$$\iint\limits_D f_n(s)f(t)dsdt = \int\limits_0^x dt \int\limits_0^{x-t} f_n(s)ds = \int\limits_0^x F_n(t)dt \quad .$$

In this case we also have $x-r \le 0$ and hence $F_n(x-r) = 0$.
3) $nr \le x \le (n+1)r$ (see Fig. 3). The integral is equal to the sum of two ones because region D has two parts (divided by the dashed line on Fig. 3).

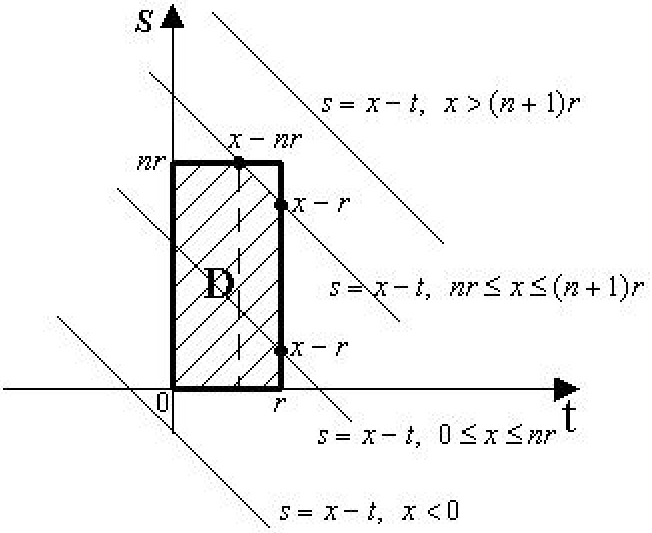

Fig. 3: Integration region for the case $x \ge nr$

$$\iint\limits_D f_n(s)f(t)dsdt = \int\limits_0^{x-nr} dt \int\limits_0^{nr} f_n(s)ds + \int\limits_{x-nr}^{r} dt \int\limits_0^{x-t} f_n(s)ds$$
$$= \left(1-(1-r)^n\right)(x-nr) + \int\limits_{x-r}^{nr} F_n(t)\,dt \;\; .$$

As a result we have the required recurrent formula for the sum distribution function:

$$F_{n+1}(x) = \begin{cases} \int\limits_0^x F_n(t)dt, \quad 0 \le x \le r \\ \int\limits_{x-r}^{x} F_n(t)dt + (1-r)F_n(x-r), \quad r \le x \le nr \\ \int\limits_{x-r}^{nr} F_n(t)dt + (x-nr) + (1-r)F_n(x-r), \quad nr \le x \le (n+1)r \end{cases}$$

□

It the next lemma it is shown that this distribution is a piece-wise function having $(n+1)$ pieces on $[0,\ (n+1)r]$ segment.

Lemma 2. *The following recurrent formula is true for the sum distribution function* $F_n(x)$*:*

$$F_{n+1}(x) = \begin{cases} F^1_{n+1}(x) = \int\limits_0^x F^1_n(t)dt, \quad 0 \le x \le r \\ F^2_{n+1}(x) = \int\limits_{x-r}^{r} F^1_n(t)dt + \int\limits_r^x F^2_n(t)dt + (1-r)F^1_n(x-r), \\ \quad r \le x \le 2r \\ \dots \\ F^k_{n+1}(x) = \int\limits_{x-r}^{(k-1)r} F^{k-1}_n(t)dt + \int\limits_{(k-1)r}^{x} F^k_n(t)dt + (1-r)F^{k-1}_n(x-r), \\ \quad (k-1)r \le x \le kr \\ \dots \\ F^n_{n+1}(x) = \int\limits_{x-r}^{(n-1)r} F^{n-1}_n(t)dt + \int\limits_{(n-1)r}^{x} F^n_n(t)dt + (1-r)F^{n-1}_n(x-r), \\ \quad (n-1)r \le x \le nr \\ F^{n+1}_{n+1}(x) = \int\limits_{x-r}^{nr} F^n_n(t)dt + (x-nr) + (1-r)F^n_n(x-r), \\ \quad nr \le x \le (n+1)r \end{cases}$$

Proof. At first we'll prove by induction that function $F_n(x)$ has n pieces on $[0,\ nr]$ segment:

$$F_n(x) = \begin{cases} F_n^1(x), & 0 \le x \le r \\ F_n^2(x), & r \le x \le 2r \\ \dots \\ F_n^n(x), & (n-1)r \le x \le nr \end{cases}$$

For $n = 1$ this statement is true: $F_1(x) = F_1^1(x) = x$ for $0 \le x \le r$. Assume that it is true for $F_n(x)$ and prove that it is also true for $F_{n+1}(x)$. We'll use the recurrent formula for $r \le x \le nr$:

$$F_{n+1}(x) = \int\limits_{x-r}^{x} F_n(t)dt + (1-r)F_n(x-r) \ \ .$$

As soon as $F_n(x)$ has n pieces $F_n^1, ..., F_n^n$ then on every segment $[(k-1)r,\ kr]$ (where $k = \overline{2,n}$) $\quad F_{n+1}(x)$ has different expressions. If $x \in [(k-1)r,\ kr]$, then:

$$\begin{aligned} x - r \in [(k-2)r,\ (k-1)r] \ \ , \\ F_n(x-r) = F_n^{k-1}(x-r) \ \ , \\ \int\limits_{x-r}^{x} F_n(t)dt = \int\limits_{x-r}^{(k-1)r} F_n^{k-1}(t)dt + \int\limits_{(k-1)r}^{x} F_n^k(t)dt \ \ . \end{aligned}$$

Thus for $[(k-1)r,\ kr]$ segment we have:

$$F_{n+1}(x) = F_{n+1}^k(x) = \int\limits_{x-r}^{(k-1)r} F_n^{k-1}(t)dt + \int\limits_{(k-1)r}^{x} F_n^k(t)dt + (1-r)F_n^{k-1}(x-r) \ \ .$$

For $[0,\ r]$ segment:

$$F_{n+1}(x) = F_{n+1}^1(x) = \int\limits_{0}^{x} F_n^1(t)dt \ \ .$$

And for $[nr,\ (n+1)r]$ segment:

$$F_{n+1}(x) = \int\limits_{x-r}^{nr} F_n(t)dt + (x - nr) + (1-r)F_n(x-r) \ \ ,$$

$$F_{n+1}(x) = F_{n+1}^{n+1}(x) = \int\limits_{x-r}^{nr} F_n^n(t)dt + (x - nr) + (1-r)F_n^n(x-r) \ \ .$$

This proves our statement for $F_{n+1}(x)$. The desired recurrent formula follows immediately from this proof. $\square$

This formula will be used later to get the main formula for $F_n(x)$, but before we need to derive an auxiliary formula for $F_n^n(x)$.

2.2 Auxiliary Formula

Lemma 3. *The following formula is true for* $F_n(x)$ *on* $[(n-1)r,\ nr]$ *segment:*

$$F_n^n(x) = 1 - (-1)^n \sum_{i=0}^{n} C_n^i \, (r-1)^i \, \frac{(x-nr)^{n-i}}{(n-i)!} \quad .$$

Proof. Let us prove this formula by induction. For $n = 1$ it is true:

$$F_1^1(x) = 1 - (-1)^1 \left((r-1)^0 \, \frac{(x-r)^1}{1!} + (r-1)^1 \, \frac{(x-r)^0}{0!} \right) = x \quad .$$

Assuming that the formula is true for F_n^n we'll show that it is also true for F_{n+1}^{n+1} using the recurrent relation:

$$F_{n+1}^{n+1}(x) = (x-nr) + \int_{x-r}^{nr} F_n^n(t)dt + (1-r)F_n^n(x-r) \quad ,$$

$$F_n^n(x) = 1 - (-1)^n \sum_{i=0}^{n} C_n^i \, (r-1)^i \, \frac{(x-nr)^{n-i}}{(n-i)!} \quad .$$

After inserting the expression for F_n^n, taking the integral and changing the index variable in the last sum we obtain:

$$\begin{aligned} F_{n+1}^{n+1}(x) = (x-nr) + nr - (x-r) - 0 + (-1)^n \sum_{i=0}^{n} C_n^i V_i \\ + (1-r) + (-1)^n \sum_{i=1}^{n+1} C_n^{i-1} V_i \quad , \end{aligned}$$

where

$$V_i = (r-1)^i \, \frac{(x-(n+1)r)^{n+1-i}}{(n+1-i)!} \quad .$$

And finally, using the relation $C_n^i + C_n^{i-1} = C_{n+1}^i$, we get the desired expression for F_{n+1}^{n+1}:

$$F_{n+1}^{n+1}(x) = 1 - (-1)^{n+1}\left(\sum_{i=1}^{n} C_{n+1}^{i} V_i + V_0 + V_{n+1}\right)$$

$$= 1 - (-1)^{n+1}\sum_{i=0}^{n+1} C_{n+1}^{i}\,(r-1)^{i}\,\frac{(x-(n+1)r)^{n+1-i}}{(n+1-i)!} \quad .$$

□

Now everything is ready for the main formula proof.

2.3 Main Formula

Theorem 1. *The following formula is true for* $F_n(x)$ *on* $[(k-1)r,\ kr]$ *segment* ($k = \overline{1,n}$)*:*

$$F_n^k(x) = \sum_{i=0}^{k-1}\left[(-1)^i C_n^I \sum_{j=0}^{i} C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j}}{(n-j)!}\right] \quad .$$

To simplify the expressions used in the proof we'll introduce the following denotations:

$$V_{i,\,j}(x) = (r-1)^j\,\frac{(x-ir)^{n-j+1}}{(n-j+1)!} \quad ,$$

$$U_i(x) = (-1)^i \sum_{j=0}^{i} C_i^j V_{i,\,j}(x) = (-1)^i \sum_{j=0}^{i} C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j+1}}{(n-j+1)!} \quad .$$

The proof is divided into several parts.

Lemma 4. *The formula from theorem 1 is true for* $k = n$*.*

Proof. We'll prove the formula for $F_n^n(x)$ by induction. For $n = 1$ it is true:

$$F_1^1(x) = (-1)^0\,(r-1)^0\,\frac{x^1}{1!} = x \quad .$$

Assume that formula is true for $F_n^n(x)$:

$$F_n^n(x) = \sum_{i=0}^{n-1}\left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j}}{(n-j)!}\right] \quad . \tag{1}$$

Also we have an auxiliary formula (lemma 3):

$$F_n^n(x) = 1 - (-1)^n \sum_{j=0}^{n} C_n^j\,(r-1)^j\,\frac{(x-nr)^{n-j}}{(n-j)!} \quad . \tag{2}$$

Subtracting these two equalities (1)-(2) we obtain:

$$1 = \sum_{i=0}^{n} \left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j \, (r-1)^j \, \frac{(x-ir)^{n-j}}{(n-j)!} \right] \ . \tag{3}$$

Again to prove the formula for $F_{n+1}^{n+1}(x)$ we apply the recurrent formula from lemma 2:

$$F_{n+1}^{n+1}(x) = (x-nr) + \int_{x-r}^{nr} F_n^n(t)dt + (1-r)F_n^n(x-r) \ . \tag{4}$$

Replacing $(x-nr)$ with $\int_{nr}^{x} 1 \, dt$ and inserting the expression (3) instead of 1 into this integral, we have:

$$\begin{aligned} (x-nr) = \int_{nr}^{x} \left[\sum_{i=0}^{n} \left((-1)^i C_n^i \sum_{j=0}^{i} C_i^j \, (r-1)^j \, \frac{(t-ir)^{n-j}}{(n-j)!} \right) \right] dt \\ = \sum_{i=0}^{n} C_n^i U_i(x) - \sum_{i=0}^{n} C_n^i U_i(nr) \ . \end{aligned}$$

Inserting the expression (1) for $F_n^n(x)$ to the second and third items of the recurrent formula (4), we have:

$$\begin{aligned} \int_{x-r}^{nr} F_n^n(t)dt &= \sum_{i=0}^{n-1} C_n^i U_i(nr) - \sum_{i=0}^{n-1} \left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j V_{i+1,j}(x) \right] \ , \\ (1-r)F_n^n(x-r) &= -\sum_{i=0}^{n-1} \left[(-1)^i C_n^i \sum_{j=1}^{i+1} C_i^{j-1} V_{i+1,j}(x) \right] \ . \end{aligned}$$

Note that $U_n(nr) = 0$ and thus

$$\sum_{i=0}^{n} C_n^i U_i(nr) = \sum_{i=0}^{n-1} C_n^i U_i(nr) \ .$$

So summing all the three items of (4) we obtain:

$$\begin{aligned} F_{n+1}^{n+1}(x) = \sum_{i=0}^{n} C_n^i U_i(x) - \sum_{i=0}^{n-1} \left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j V_{i+1,j}(x) \right] \\ - \sum_{i=0}^{n-1} \left[(-1)^i C_n^i \sum_{j=1}^{i+1} C_i^{j-1} V_{i+1,j}(x) \right] \ . \end{aligned}$$

Joining the last two sums into one and using relation $C_i^j + C_i^{j-1} = C_{i+1}^j$, we have:

$$F_{n+1}^{n+1}(x)=\sum_{i=0}^{n}C_n^iU_i(x)-\sum_{i=0}^{n-1}\left[(-1)^iC_n^i\left(\sum_{j=1}^{i}C_{i+1}^jV_{i+1,j}(x)+V_{i+1,0}(x)+V_{i+1,i+1}(x)\right)\right]$$
$$=\sum_{i=0}^{n}C_n^iU_i(x)-\sum_{i=0}^{n-1}\left[(-1)^iC_n^i\sum_{j=0}^{i+1}C_{i+1}^jV_{i+1,j}(x)\right]=\sum_{i=0}^{n}C_n^iU_i(x)+\sum_{i=1}^{n}C_n^{i-1}U_i(x)\ .$$

Finally making some transformations with these two sums, joining them into one and using relation $C_n^i+C_n^{i-1}=C_{n+1}^i$, we get the required expression:

$$F_{n+1}^{n+1}(x)=U_0(x)+\sum_{i=1}^{n}C_{n+1}^iU_i(x)=\sum_{i=0}^{n}C_{n+1}^i\,U_i(x)$$
$$=\sum_{i=0}^{n}(-1)^iC_{n+1}^i\sum_{j=0}^{i}C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j+1}}{(n-j+1)!}\ .$$

□

The next special case to be proved for the main formula is $k=1$.

Lemma 5. *The formula from theorem 1 is true for* $k=1$.

Proof. The case $k=1$ is the simplest one, and the formula to be proved is the following:

$$F_n^1=\frac{x^n}{n!}$$

For $n=1$ it is true:

$$F_1^1=\frac{x^1}{1!}=x$$

Assuming it is true for F_n^1, we show that it is also true for F_{n+1}^1 by means of the recurrent relation from lemma 2:

$$F_{n+1}^1=\int_0^x F_n^1(t)dt=\int_0^x\frac{t^n}{n!}dt=\frac{x^{n+1}}{(n+1)!}\ .$$

□

Now we are going to consider the general case for the main formula: $k=2,\ 3,\ ...,\ n-1$ and prove the theorem.

Proof. For proving of the main formula for $k=2,\ 3,\ ...,\ n-1$ we again use induction by n. The formula is true for $n=1,2$ because for F_1^1,F_2^1 it satisfies lemma 5 and for F_2^2 it satisfies lemma 4. Now assuming that the formula is true for F_n^k, $k=1,\ 2,\ 3,\ ...,\ n$ it is necessary to prove it for F_{n+1}^k, $k=2,\ 3,\ ...,\ n-1$. So from this assumption and from the recurrent formula we have:

$$F_n^k(x)=\sum_{i=0}^{k-1}\left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j}}{(n-j)!}\right] \quad ,$$

$$F_n^{k-1}(x)=\sum_{i=0}^{k-2}\left[(-1)^i C_n^i \sum_{j=0}^{i} C_i^j\,(r-1)^j\,\frac{(x-ir)^{n-j}}{(n-j)!}\right] \quad ,$$

$$F_{n+1}^k(x)=\int\limits_{x-r}^{(k-1)r} F_n^{k-1}(t)dt+\int\limits_{(k-1)r}^{x} F_n^k(t)dt+(1-r)F_n^{k-1}(x-r) \quad .$$

The items of the sum in the recurrent relation can be rewritten as:

$$\int\limits_{x-r}^{(k-1)r} F_n^{k-1}(t)dt=\sum_{i=0}^{k-2} C_n^i U_i(t)\Big|_{x-r}^{(k-1)r}$$

$$=\sum_{i=0}^{k-2} C_n^i U_i\left((k-1)r\right)-\sum_{i=0}^{k-2}(-1)^i C_n^i \sum_{j=0}^{i} C_i^j V_{i+1,j}(x) \quad ,$$

$$\int\limits_{(k-1)r}^{x} F_n^k(t)dt=\sum_{i=0}^{k-1} C_n^i U_i(t)\Big|_{(k-1)r}^{x}=\sum_{i=0}^{k-1} C_n^i U_i(x)-\sum_{i=0}^{k-1} C_n^i U_i\left((k-1)r\right) \quad ,$$

$$(1-r)\,F_n^{k-1}(x-r)=-\sum_{i=0}^{k-2}(-1)^i C_n^i \sum_{j=1}^{i+1} C_i^{j-1} V_{i+1,j}(x) \quad .$$

Note, that $U_{k-1}\left((k-1)r\right)=0$ and hence

$$\sum_{i=0}^{k-1} C_n^i U_i\left((k-1)r\right)=\sum_{i=0}^{k-2} C_n^i U_i\left((k-1)r\right) \quad .$$

So after summation we obtain:

$$F_{n+1}^k(x)=-\sum_{i=0}^{k-2}(-1)^i C_n^i \sum_{j=0}^{i} C_i^j V_{i+1,j}(x)+\sum_{i=0}^{k-1} C_n^i U_i(x)$$
$$-\sum_{i=0}^{k-2}(-1)^i C_n^i \sum_{j=1}^{i+1} C_i^{j-1} V_{i+1,j}(x) \quad .$$

Joining of the first and the last sum and applying relation $C_i^j+C_i^{j-1}=C_{i+1}^j$ gives us:

$$
\begin{aligned}
F^k_{n+1}(x) = \sum_{i=0}^{k-1} C_n^i U_i(x) - \sum_{i=0}^{k-2} (-1)^i C_n^i \left(\sum_{j=1}^{i} C_{i+1}^j V_{i+1,j}(x) + V_{i+1,0} + V_{i+1,i+1} \right) \\
= \sum_{i=0}^{k-1} C_n^i U_i(x) + \sum_{i=0}^{k-2} C_n^i U_{i+1}(x) \ .
\end{aligned}
$$

Finally these two sums are also joined and relation $C_n^i + C_n^{i-1} = C_{n+1}^i$ is applied after some simple transformations:

$$
F^k_{n+1}(x) = U_0(x) + \sum_{i=1}^{k-1} C_{n+1}^i U_i(x) = \sum_{i=0}^{k-1} \left[(-1)^i C_{n+1}^i \sum_{j=0}^{i} C_i^j \, (r-1)^j \, \frac{(x-ir)^{n+1-j}}{(n+1-j)!} \right] \ .
$$

This completes the induction and proves the main formula for all n and k. □

3 Acknowledgement

The authors are partially supported by LATNA Laboratory, NRU HSE, RF government grant, ag.11.G34.31.0057.

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