# Kazhdan-Lusztig-polynomials and indecomposable bimodules over polynomial rings

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January 9, 2006

**Abstract**

We develop a strategy to prove the positivity of coefficient of Kazhdan-Lusztig polynomials for arbitrary Coxeter groups. Thanks go to Martin H"arterich and Catharina Stroppel for corrections to previous versions. In particular I thank Patrick Polo, whose Remarks led to a sustancial improvement of this article.

## Contents

# 1 Realization of Hecke algebras via bimodules

*Notation* 1.1. Given a svelte additive categary $\mathcal{A}$ denote its split Grothendieck-group, thus the free abelian group over the objects with relations $M = M' + M''$ whenever we have $M \cong M' \oplus M''$. Given a object $M$ let $\langle M \rangle$ denote its class in $\langle \mathcal{A} \rangle$.

*Notation* 1.2. Given a $\mathbb{Z}$-graded ring $A$ let $A\text{-mod}^{\mathrm{f}}_{\mathbb{Z}}$ denote the category of all finitely generated $\mathbb{Z}$-graded $A$-modules. We write $M[n]$ for the object $M$ with its $\mathbb{Z}$-grading shifted by $n$, in formulas $(M[n])_i = M_{i+n}$.

*Remark* 1.3. Given a field $k$ and a finitely generated commutative nonnegatively $\mathbb{Z}$-graded $k$-algebra $A$ with $A_0 = k$ the endomorphism ring of a finitely $\mathbb{Z}$-graded $A$-module is always of finite dimension. The endomorphism ring of an indecomposable such module has no indempotents except zero and one and thus is indecomposable as a right module over itself. But it is also the endomorphism ring of this indecomposable right module of finite length and is thus local. Hence we have in $A\text{-mod}^{\mathrm{f}}_{\mathbb{Z}}$ the Krull-Schmid theorem with the same proof as in [Pie82] 5.4 and the isomorphy classes of indecomposable objects form a basis of $\langle A\text{-mod}^{\mathrm{f}}_{\mathbb{Z}} \rangle$, so in particular we have $\langle N \rangle = \langle M \rangle \Leftrightarrow N \cong M$.

*Notation* 1.4. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with a finite number of generators, $|\mathcal{S}| < \infty$. Let $l : \mathcal{W} \to \mathbb{N}$ be its length function and $\leq$ the Bruhat order on $\mathcal{W}$. In particular $x < y$ means $x \leq y$, $x \neq y$. On the free $\mathbb{Z}[v, v^{-1}]$-mdule

$$\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S}) = \bigoplus_{x \in \mathcal{W}} \mathbb{Z}[v, v^{-1}] T_x$$

over $\mathcal{W}$ there is exactly one structure of associative $\mathbb{Z}[v, v^{-1}]$-algebra with $T_x T_y = T_{xy}$ if $l(x) + l(y) = l(xy)$ and $T_s^2 = v^{-2} T_e + (v^{-2} - 1) T_s$ for all $s \in \mathcal{S}$, see [Bou81], IV, §2, Exercice 23. This associative algebra $\mathcal{H}$ is called the Hecke algebra of $(\mathcal{W}, \mathcal{S})$. It is unitary with unit $T_e$, we abreviate often $T_e = 1$. In fact it might be more natural to use $q = v^{-2}$, and the first sections of this article would gain clarity thereby. In the later sections however it is essential for a transparent exposition to have a root of $q$ at our disposal. wichtig, eine Wurzel von $q$ zur Verf"ugung zu haben. The Hecke algebra can also be described as the assocative unitary $\mathbb{Z}[v, v^{-1}]$ algebra with generators $\{T_s\}_{s \in \mathcal{S}}$, the quadratic relations $T_s^2 = v^{-2} T_e + (v^{-2} - 1) T_s$ and the so-called braid relations $T_s T_t \ldots T_s = T_t T_s \ldots T_t$ resp. $T_s T_t T_s \ldots T_t = T_t T_s T_t \ldots T_s$ in case $st \ldots s = ts \ldots t$ resp. $sts \ldots t = tst \ldots s$ for $s, t \in \mathcal{S}$.

**Definition 1.5.** Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. Let $\mathcal{T} \subset \mathcal{W}$ denote the set of all "reflections", i.e. all elements of $\mathcal{W}$, which are conjugate to elements of $\mathcal{S}$. By a reflection faithful representation of our Coxeter system we mean a representation $\mathcal{W} \hookrightarrow \mathrm{GL}(V)$ in a finite dimensional vector space over a field $k$ of characteristic $\operatorname{char} k \neq 2$ with the following properties:

1. Our representation is faithful.
2. For $x \in \mathcal{W}$ we have $\dim(V/V^x) = 1 \Leftrightarrow x \in \mathcal{T}$. Thus exactly the reflections from $\mathcal{W}$ have a fixed point set of codimension one in $V$.

*Remark* 1.6. Given a reflection faithful representation the elements of $\mathcal{T}$ are precisely those elements of $\mathcal{W}$, which act as reflections on $V$, thus which decompose in $V$ in a onedimensional eigenspace of eigenvalue $-1$ and a hyperplane of invariand vectors. Indeed $x \in \mathcal{T}$ cannot act unipotently, since an automorphism of order two of a vector space over a field of charcteristic $\neq 2$ is diagonalizable and our condition excludes that $x$ acts as the identity. Furthermore reflections from $\mathcal{W}$ can be distinguished as well by their eigenspaces of eigenvalue $(-1)$ as well as by their reflecting hyperplanes, so that for $t, r \in \mathcal{T}$ we have

$$V^t = V^r \quad \Leftrightarrow \quad V^{-t} = V^{-r} \quad \Leftrightarrow \quad t = r.$$

For example in case $G = V^{-t} = V^{-r}$, we consider the short exact sequence $G \hookrightarrow V \twoheadrightarrow V/G$. Since both $t$ and $r$ act as the identity on $V/G$ and since taking the adjoint map doesn't change the dimension of the eigenspaces we see that $tr$ fixes a hyperplane. But since $tr$ has determinant 1 it must by condition 2 act as the identity on $V$ and thus by condition 2 is the identity of $\mathcal{W}$. We show in the next section, that the representation of a Coxeter group generalizing the representation of an affine Weyl group on the Cartan of a Kac-Moody algebra is always reflection faithful.

**Definition 1.7.** A representation of $\mathcal{W}$, in which reflections act as reflections and where different reflections have different $(-1)$-eigenspaces will be called reflection vector faithful.

*Example* 1.8. The geometric representation of an infinite dieder group is reflection vector faithful. However it is not faithful, since all elements expect the neutral element have a fixed point set of codimension one, not only the reflections.

*Notation* 1.9. Let now for the sake of simplicity of notation $k$ be an infinite field and let $R = R(V)$ donote the $k$-algebra of all regular functions on the space $V$ underlying a reflection vector faithful representation. We equip $R$

with a $\mathbb{Z}$-grading $R = \bigoplus_{i\in\mathbb{Z}} R_i$ such, that we have $R_2 = V^*$ and $R_i = 0$ for uneven $i$. Again it would be more natural here to work with the usual grading, but in the long run we need these conventions. Let

$$\mathcal{R} = \mathcal{R}_V \subset R\text{-mod}_{\mathbb{Z}}\text{-} R$$

denote the categary of all $\mathbb{Z}$-graded $R$-bimodules, which are finitely generated from the right as well as from the left, and where the action of $k$ from the right and the left is the same. Tensoring $\otimes_R$ makes

$$\langle \mathcal{R} \rangle$$

into a ring. For $s \in \mathcal{S}$ let $R^s \subset R$ denote the subring of all $s$-invariants.

**Theorem 1.10.** *Let $V$ be a reflection vector faithful representation of the Coxeter system $(\mathcal{W}, \mathcal{S})$ over an infinite field. Let $\mathcal{H}$ be the Hecke algebra and $R$ the ring of polynomial functions on $V$. There is exactely one ring homomorphism*

$$\mathcal{E} : \mathcal{H} \to \langle \mathcal{R} \rangle$$

*such that we have $\mathcal{E}(v) = \langle R[1] \rangle$ and $\mathcal{E}(T_s + 1) = \langle R \otimes_{R^s} R \rangle \quad \forall s \in \mathcal{S}$.*

*Remark* 1.11. This theorem is a variant of theorem 1 from [Soe92]. I prefer the proof given here, since it does not assume any knowledge of Demazure operators.

*Proof.* Unicity is evident, since $\mathcal{H}$ is generated by $T_s + 1$ as an algebra over $\mathbb{Z}[v, v^{-1}]$. Existence needs only to be proven in case of a dieder group, since $\mathcal{H}$ can be described by relations involving each time only two generators. The dieder case will be discussed in section 4. □

**Definition 1.12.** We consicer in the Hecke algebra the elements $\tilde{T}_x = v^{l(x)} T_x$. By [KL79] there is exactly one involution $d : \mathcal{H} \to \mathcal{H}$ with $d(v) = v^{-1}$ and $d(T_x) = (T_{x^{-1}})^{-1}$ and for $x \in \mathcal{W}$ there is a unique $C'_x \in \mathcal{H}$ with $d(C'_x) = C'_x$ and

$$C'_x \in \tilde{T}_x + \sum_y v\mathbb{Z}[v]\tilde{T}_y$$

These elements form the so-called Kazhdan-Lusztig-basis of the Hecke algebra from [KL79].

*Conjecture* 1.13. Let $V$ be a reflection faithful representation of $\mathcal{W}$ over an infinite field ans let $R$ be the ring of regular functions on $V$. Then there is, at least in case $k = \mathbb{C}$, an indecomposable $\mathbb{Z}$-graded $R$-bimodule $B_x \in \mathcal{R}$ such that

$$\mathcal{E}(C'_x) = \langle B_x \rangle.$$

*Remark* 1.14. In case $k = \mathbb{C}$ and $\mathcal{W}$ a finite Weyl group this is shown in [Soe92]. For $\mathcal{W}$ a finite Weyl group and $\operatorname{char} k$ at least the Coxeter number it is proven in [Soe00], that this conjecture is equivalent to a part of Lusztig's conjecture concerning characters of irreducible representations of algebaric groups over $k$. For the case of the action of a Weyl group on the Cartan of an affine Kac-Moody algebra the conjecture is established in [Här99]. In 5.3 we construct a left inverse to $\mathcal{E}$, whose explicite description shows, how the preceding conjecture implies the positivity of all coefficients of all Kazhdan-Lusztig polynomials.

*Remark* 1.15. In 6.16 we show, that the "indecomposable bimodules in the image of $\mathcal{E}$" can at least be parametrized by $\mathcal{W}$. Furthermore we determine in 5.15 the dimension of the Hom-spaces between any two "bimodules im the image of $\mathcal{E}$" and show in which way our formulas are compatible with conjecture 1.13.

*Remark* 1.16. Still open is in particular the case of universal Coxeter groups (here the KL-polynomials were described by [Dye88]) and the case op general finite Coxeter groups, which should be accessible by computer.

*Remark* 1.17. Related investigations are in [BM01] and in partially unpublished work of Dyer [Dye94] and in unpublished work of Peter Fiebig.

# 2 A reflection faithful representation

**Proposition 2.1.** *Given a finite set $\mathcal{S}$ and a Coxete rmatrix of type $\mathcal{S}$ of the formbe a finite dimensional real vector space and let be given linearly independent vectors $(e_s)_{s\in\mathcal{S}}$ in $V$ and linearly independent linear forms $(e_s^\vee)_{s\in\mathcal{S}}$ on $V$ such that we have*

$$\langle e_t, e_s^\vee\rangle = -2\cos(\pi/m_{s,t}) \quad \forall s,t \in \mathcal{S}.$$

*If we assume as in [Kac90] that the dimension of $V$ is smallest possible, then the formula $\rho(s)(v) = v - \langle v, e_s^\vee\rangle e_s$ defines a reflection faithful representation $\rho : \mathcal{W} \to \mathrm{GL}(V)$ of our Coxeter group.*

*Proof.* That we get in this way a repesentation, i.e. that the braid relations are satisfied can be deduced from the fact that for any pair $s, t$ of simple reflections with $st$ of finite order our space decomposes as the intersection of the kernels of $e_s^\vee$ and $e_t^\vee$, which is not moved by $s$ and $t$ at all, and the space generated by $e_s$ and $e_t$, where $s$ and $t$ act as generators of the usual dieder group. In general the subspace $E \subset V$ spanned by the $e_s$ is a subrepresentation isomorphic to the "natural representation" of [Bou81], V.4.3 and is in particular faithful. Furthermore $\mathcal{W}$ acts trivially on the simultaneous kernel $K$ of all $e_s^\vee$ and the natural action on $(V/K)^*$ is isomorphic again to the natural representation, which means that the action on $(V/K)$ is isomorphic to the contragredient representation $E^*$ of [Bou81], V.4.4. In view of our minimality assumption on the dimension of $V$ we certainly have $K \subset E$ and thus a filtration by subrepresentations

$$0 \subset K \subset E \subset V.$$

The pairing $(e_s, e_t) = \frac{1}{2}\langle e_s, e_t^\vee\rangle$ now defines an invariant symetric form on $E$ and a nondegenerate invariant symmetric form on $E/K$. Now let $x \in \mathcal{W}$ be an element, whose fix point set $V^x \subset V$ is a hyperplane. We have to show $x \in \mathcal{T}$. Let $X \in \operatorname{End} V$ denote the image of $x$. We have either $\det X = 1$ or $\det X = -1$ and start treating the case $\det X = 1$. If we choose a complement $\mathbb{R}c$ of the hyperplane $V^x$, then our endomorphism $X$ necessarily has the form $X(v+\alpha c) = v+\alpha v_0+\alpha c$ for some fixed $v_0 \in V^x$. Certainly we have $V^x \supset K$. Since our group acts faithfully on $E$ we can choose $c \in E$ and deduce $v_0 \in E$. Since $E^* \cong V/K$ is a faithful representation as well, we get $v_0 \notin K$. Now we use our invariant bilinear form on $E$ and deduce $(v + \alpha c, v + \alpha c) = (v+\alpha v_0+\alpha c, v+\alpha v_0+\alpha c)$ alias $2\alpha(v, v_0)+\alpha^2(2(c, v_0)+(v_0, v_0)) = 0$ for all $v \in E$ and $\alpha \in \mathbb{R}$. This gives us $(v, v_0) = 0$ for all $v \in E$ contrdicting $v_0 \notin K$. Thus the case $\det X = 1$ is not possible and we can assume $\det X = -1$. Then necessarily $x$ acts as a reflection on $V$ and also on $V/K \cong E^*$. Since $\mathcal{W}$ acts faithfully and transitively on the alcoves of the Tits cone, the fix point set of $x$ cannot meet any alcove in the Tits cone. If we consider the fundamental dominant alcove of the Tits cone $C \subset E^*$ and its image under $x$, we find that $C$ and $xC$ are separated only by finite number of reflecting hyperplanes $V^z$ for some $z \in \mathcal{T}$. On the other hand a certain nonempty open part of the refection plane of $x$ in $E^*$ consits of points on segments joining points of $C$ with points of $xC$, consider for example the image of $C$ under the projection $(\mathrm{id}+x)/2$. This open part then has to be covered by our finite number of reflecting hyperplans, and this is impossible unless the reflecting hyperplane of $x$ coincides with the reflecting hyperplane of some $z \in \mathcal{T}$. But then we form $zx$ and come back to the impossible case $\det = 1$ least we have $x = z$. □

# 3 Further notations and formulas

*Notation* 3.1. Given any finite dimensional representation $V$ of a group $\mathcal{W}$ we consider for any $x \in \mathcal{W}$ the (reversed) graph

$$\mathrm{Gr}(x) = \{(x\lambda, \lambda) \mid \lambda \in V\} \subset V \times V$$

and form for any finite subset $A \subset \mathcal{W}$ in $V \times V$ the Zariski closed subset

$$\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x).$$

*Remark* 3.2. For any $y, z \in \mathcal{W}$ we get obviously an isomorphic

$$\mathrm{Gr}(y) \cap \mathrm{Gr}(z) \xrightarrow{\sim} V^{yz^{-1}}$$

by the projection on the first coordinate and we have

$$(\mathrm{Gr}(y) + \mathrm{Gr}(z)) \cap (V \times 0) = \mathrm{im}(yz^{-1} - \mathrm{id}) \times 0.$$

*Notation* 3.3. Let now $k$ be an infinite field and let as before denote $R$ the $k$-algebra of all regular functions on $V$. We always abreviate $\otimes_k = \otimes$. If we identify $R \otimes R$ with the regular functions on $V \times V$ via the rule $(f \otimes g)(\lambda, \mu) = f(\lambda)g(\mu)$, the regular functions on $\mathrm{Gr}(A)$ as a quotient of $R \otimes R$ from naturally a $\mathbb{Z}$-graded $R$-bimodul. This $\mathbb{Z}$-graded $R$-bimodul we denote

$$R(A) = R(\mathrm{Gr}(A)) \in R\text{-mod}_{\mathbb{Z}}\text{-}R$$

and it is easy to check that it is finitely generated as a left module and as a right module over $R$. For $A = \{x_1, \ldots, x_n\}$ we also put $\mathrm{Gr}(A) = \mathrm{Gr}(x_1, \ldots, x_n)$ and $R(A) = R(x_1, \ldots, x_n)$ or even shorter $R(x) = R_x$, $R(x, y) = R_{x,y}$. In the case $A = \{y \mid y \leq x\}$ we put $R(A) = R(\leq x)$. For the right action of $\mathcal{W}$ on $R$ we use exponential notation $\mathrm{S}r^y(\lambda) = r(y\lambda)$ $\forall y \in \mathcal{W}$, $\lambda \in V$, $r \in R$. If $1_y \in R_y$ denotes the constant function 1 on $\mathrm{Gr}(y)$ alias the image of $1 \otimes 1$ in $R_y$, we have $r1_y = 1_y r^y$. The notations are chosen to get $R(x) \otimes_R R(y) \cong R(xy)$ for all $x, y \in \mathcal{W}$.

# 4 The case of a dieder group

*Remark* 4.1. If $\mathcal{W}$ is a dieder group, by [Hum90, 7.12(a)] the elementes of the selfdual basis are of the form $C'_x = v^{l(x)} \sum_{y \leq x} T_y$. We prove this as Remark 4.4.

**Theorem 4.2.** *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with twoe generators $|\mathcal{S}| = 2$ and let $V$ be a reflection vector faithful representation over an infinite field. Then the homomorphism of additive groups*

$$\begin{array}{rccl} \mathcal{E}: & \mathcal{H} & \to & \langle \mathcal{R} \rangle \\ & v^n C'_x & \mapsto & \langle R(\leq x) \rangle [n + l(x)] \end{array}$$

*is a ring homomorphism.*

*Remark* 4.3. For any reflection $s : V \to V$ the obvious map $R \otimes R \twoheadrightarrow R(e,s)$ induces an isomorphism $R \otimes_{R^s} R \xrightarrow{\sim} R(e,s)$. To see this, Remark, that this map certainly is a surjection and compare the dimensions of homogeneous parts, where one may use the eigenspace decomposition $R = R^s \oplus R^s\alpha$ for $\alpha \in V^*$ an equation of the reflection plane of $s$ and a short exact sequence $R_s[-2] \hookrightarrow R(e,s) \twoheadrightarrow R_e$. Thus the theorem indeed gives the existence in Theorem 1.10. the proof biven here extends arguments of I. Herrmann [Her99].

*Proof.* Certainly it suffices, to show for all simple reflections $s$ and all $x \in \mathcal{W}$ the equation

$$\mathcal{E}\left((T_s+1)\sum_{y\le x} T_y\right) = \mathcal{E}(T_s+1)\;\mathcal{E}\left(\sum_{y\le x} T_y\right).$$

If we put $A = \{y \in \mathcal{W} \mid y \le x\}$, then $A \cup sA$ and $A \cap sA$ are both of the form $\{y \in \mathcal{W} \mid y \le z\}$ for suitable $z$ in our dieder group $\mathcal{W}$, unless we are in the second case and the intersection is empty. An explicite computation in the Hecke algebra now shows

$$(T_s+1)\sum_{y\in A} T_y = \sum_{y\in A\cup sA} T_y \;+\; v^{-2}\sum_{y\in A\cap sA} T_y.$$

If we insert this results on the left hand side, our equation transforms into claiming an isomorphism of graded bimodules

$$R(A\cup sA) \oplus R(A\cap sA)[-2] \cong R \otimes_{R^s} R \otimes_R R(A),$$

whic will be shown after a preliminary Lemma as Proposition 4.6. □

*Remark* 4.4. To prove the formulas claimed in 4.1 for the $C'_x$ in the dieder case we only have to show the $\gamma_x = v^{l(x)}\sum_{y\le x} T_y$ are selfdual. For any simple reflection $s$ with $sx > x$ however we can rewrite one of the equations of our proof to read

$$v(T_s+1)\gamma_x = \begin{cases} \gamma_{sx} + \gamma_z \text{ with } z < x & \text{in case } l(x) > 1; \\ \gamma_{sx} & \text{in case } l(x) \le 1, \end{cases}$$

and thus the selfduality of $v(T_s+1)$ implies inductively the selfduality of all $\gamma_x$.

**Lemma 4.5.** *Let $V$ be a finite dimensional representation of a group $\mathcal{W}$ over an infinite field $k$ of charakteristc $\neq 2$. Let $A \subset \mathcal{W}$ be a finite subset and $s \in \mathcal{W}$ an element with $sA = A$, acting on $V$ as a reflection. Then we have:*

1. *There is an isomorphism of graded bimodules*

$$R \otimes_{R^s} R(A) \cong R(A) \oplus R(A)[-2].$$

2. *For* $R(A)^+ \subset R(A)$ *the invariants under the action of* $s \times \mathrm{id}$ *multiplication induces an isomorphism*

$$R \otimes_{R^s} R(A)^+ \xrightarrow{\sim} R(A).$$

*Proof.* Let first $W$ be an arbitray finite dimensional vector space over $k$. Any reflection $t : W \to W$ defines an involution $t : R(W) \to R(W)$, and if we choose an equation $\beta \in W^*$ of the reflecting hyperplane $W^t$, we can consider

$$\partial_t = \partial_t^\beta : R(W) \to R(W), \quad f \mapsto \frac{f - tf}{2\beta}.$$

If now $X \subset W$ is a Zariski closed $t$-stable subset, then $t$ induces an involution on $R(X)$ and we get an eigenspace decomposition of the form $R(X) = R(X)^+ \oplus R(X)^-$. If no irreducible component of $X$ is contained in $W^t$, our operator $\partial_t$ stabilizes the kernel of the sujection $R(W) \twoheadrightarrow R(X)$ and thus induces a map $\partial_t : R(X) \to R(X)$. It is easy to see that $\partial_t$ and multiplication by $\beta$ are mutually inverse isomorphisms $R(X)^+ \cong R(X)^-[2]$ of $\mathbb{Z}$-graded $R^t$-modules.

Now let us take $W = V \times V$ and $t = s \times \mathrm{id}$ for our refection $s \in \mathcal{W}$. We apply our results to $X = \mathrm{Gr}(A)$ and get a decomposition $R(A) = R(A)^+ \oplus R(A)^-$ as well as an isomorphism $R(A)^+ \cong R(A)^-[2]$ by multiplication with $\alpha \otimes 1 = \beta$ for $\alpha \in V^*$ an equation of the reflecting hyperplane of $s$. With $R = R^s \oplus \alpha R^s$ we get part 2, and with our decompositon $R(A) = R(A)^+ \oplus R(A)^-$ and part 2 we also get part 1. □

**Proposition 4.6.** *Let* $(\mathcal{W}, \mathcal{S})$ *be a Coxeter system with two generators* $|\mathcal{S}| = 2$ *and let* $V$ *be a reflection faithful representation of* $\mathcal{W}$ *over an infinite field. Let* $s \in \mathcal{S}$ *and* $x \in \mathcal{W}$ *be given and let* $A = \{y \in \mathcal{W} \mid y \leq x\}$ *denote the set of all elements below* $x$. *Then in* $R$-$\mathrm{Mod}_{\mathbb{Z}}$-$R$ *we have an isomorphism*

$$R \otimes_{R^s} R(A) \cong R(A \cup sA) \oplus R(A \cap sA)[-2].$$

*Proof.* The case $A = sA$ is already done by the preceding Lemma. In case $A = \{e\}$ our proposition just claims the isomorphism $R(\leq s) \cong R \otimes_{R^s} R$ of 4.3. Thus only the case $A \neq \{e\}$, $A \neq sA$ remains to be treated, thus $A = \{y \in \mathcal{W} \mid y \leq x\}$ with $x \neq e$ and $sx > x$. By explicite calculation we find $A - sA = \{x, rx\}$ for some reflection $r \in \mathcal{T}$ different from $s$. By our assumptions the $(-1)$-eigenspaces of reflections of $\mathcal{W}$ are pairwise different and span a twodimensional subrepresentation $U \subset V$. By 3.2 there exists a form $\beta \in V^* \times V^*$, which vanishes on $\mathrm{Gr}(x) + \mathrm{Gr}(rx)$ but not on $U \times 0$. We consider the submodules generated by the cosets $\overline{\beta}$ and $\overline{1}$ of $\beta$ und 1 in $R(A)$ over $R^s \otimes R$, call them $M, N$ and claim

1. $M \cong R(A \cap sA)^+[-2]$ in $R^s$-$\mathrm{mod}_{\mathbb{Z}}$-$R$;
2. $N \cong R(A \cup sA)^+$ in $R^s$-$\mathrm{mod}_{\mathbb{Z}}$-$R$;
3. $R(A) = M \oplus N$,

where the upper index + means $(s \times \mathrm{id})$ invariants. Once this is shown, the proposition follows from the preceding lemma 4.5. Thus we only need to show our three claims.

1. To show the first claim we Remark, that for three pairwise different elements $x, y, z \in \mathcal{W}$, whose lengths do not all have the same parity, we always have

$$\mathrm{Gr}(x) + \mathrm{Gr}(y) + \mathrm{Gr}(z) \supset U \times 0.$$

To show this we may without restriction assume $z = e$ and $x, y \in \mathcal{T}$. By section 3 we have $(\mathrm{Gr}(x) + \mathrm{Gr}(e)) \cap (U \times 0) = V^{-x} \times 0$ and the some for $y$. Since we assumed our representation reflection vector faithful, we have $V^{-x} \neq V^{-y}$ and therefore the inclusion. For $y \notin \{x, rx\}$ we thus have $\mathrm{Gr}(y) + \mathrm{Gr}(x) + \mathrm{Gr}(rx) \supset U \times 0$, in particular our function $\beta$ does not vanish on $\mathrm{Gr}(y)$ for $y \in A \cap sA$. Thus an element of $R(A)$ annihilates $\overline{\beta}$ iff it vanish on $\mathrm{Gr}(A \cap sA)$. We deduce that multiplication by $\overline{\beta}$ defines an isomorphism $R(A \cap sA)[-2] \xrightarrow{\sim} R(A)\overline{\beta}$. But the image of $R^s \otimes R$ in $R(A \cap sA)$ consists precisely on the $(s \times \mathrm{id})$ invariants, thus our isomorphism restricts to an isomorphism $R(A \cap sA)^+[-2] \xrightarrow{\sim} M$.

2. The $R^s \otimes R$-submodule generated by $\overline{1}$ in $R(A \cup sA)$ is certainly just $R(A \cup sA)^+$ and the restriction onto $\mathrm{Gr}(A)$ gives an injection

$$R(A \cup sA)^+ \hookrightarrow R(A).$$

This proves the second claim.

3. We first show $R(A) = M + N$. If $\alpha \in V^*$ denotes an equation of the reflecting plane $V^s$, then we have $R = R^s \oplus \alpha R^s$, thus $R \otimes R$ is generated as an $R^s \otimes R$-modul by $1 \otimes 1$ und $\alpha \otimes 1$. It follows that it is also generated by $1 \otimes 1$ and an arbitray not $(s \times \mathrm{id})$ invariants $\beta \in V^* \times V^*$. Our $\beta$ however cannot be invariant under $(s \times \mathrm{id})$, because otherwise ist would vanish on $\mathrm{Gr}(sx)$ and thus by the inclusion of the proof of claim 1 on all of $U \times 0$. This shows $R(A) = M + N$. We only need in addition $M \cap N = 0$. This can be seen

as follows: For $y \in A \cap sA$ the restriction on $\mathrm{Gr}(y, sy) = \mathrm{Gr}(y) \cup \mathrm{Gr}(sy)$ of any element of $N$ is invariant under $s \times \mathrm{id}$. Thus it suffices to show, that the restriction of an element of $M$ on such a $\mathrm{Gr}(y, sy)$ can only be invariant under $s \times \mathrm{id}$, if it vanishes on $\mathrm{Gr}(y, sy)$. It will be sufficient for that to show, that the restriction of $\beta$ to $\mathrm{Gr}(y, sy)$ is not invariant under $s \times \mathrm{id}$ or also, that the restriction of $\beta$ to $\mathrm{Gr}(y) + \mathrm{Gr}(sy)$ is not invariant under $s \times \mathrm{id}$. But the $s \times \mathrm{id}$ invariant forms on $\mathrm{Gr}(y) + \mathrm{Gr}(sy)$ have to vanish on the subspace $V^{-s} \times 0$ and $(\mathrm{Gr}(x) + \mathrm{Gr}(rx))$ meets $U \times 0$ in the different subspace $V^{-r} \times 0$. □

# 5 Back from bimodules to the Hecke algebra

*Notation* 5.1. We work from now on always with a fixed reflection faithful representation of a Coxeter system over an infinite field and want in this section give an explicite left inverse for our ring homomorphism $\mathcal{E} : \mathcal{H} \to \langle \mathcal{R} \rangle$. We start with some notations. Given $B, B' \in \mathcal{R}$ we set

$$\mathrm{Hom}(B, B') = \mathrm{Hom}_{R \otimes R}(B, B') \in \mathcal{R}.$$

It is understood, that the left resp. right action on $R$ on our Hom-space comes from the left resp. right action on $B$ or equivalently on $B'$, in formulas $(rf)(b) = f(rb) = r(f(b))$, $(fr)(b) = f(br) = (f(b))r$, $\forall r \in R$, $b \in B$, $f \in \mathrm{Hom}(B, B')$.

*Notation* 5.2. Given a finite dimensional $\mathbb{Z}$-graded vector space $V = \bigoplus V_i$ we define its graded dimension by

$$\underline{\dim} V = \sum (\dim V_i) v^{-i} \in \mathbb{Z}[v, v^{-1}]$$

and for a finitely generated $\mathbb{Z}$-graded right $R$-module $M$ we define its graded rank by the rule

$$\underline{\mathrm{rk}} M = \underline{\dim}(M / M R_{>0}) \in \mathbb{Z}[v, v^{-1}].$$

We thus have $\underline{\dim}(V[1]) = v(\underline{\dim} V)$ and $\underline{\mathrm{rk}}(M[1]) = v(\underline{\mathrm{rk}} M)$. Certainly our graded rank is only a reasonable notation for free modules, an we will only use it for such. By $\overline{\underline{\mathrm{rk}}} M$ we denote the image of $\underline{\mathrm{rk}} M$ under the substitution $v \mapsto v^{-1}$. The following theorem motivates this whole section.

**Theorem 5.3.** *Our map $\mathcal{E} : \mathcal{H} \to \langle \mathcal{R} \rangle$ has as a left inverse the map $\langle \mathcal{R} \rangle \to \mathcal{H}$ given by the rule*

$$\langle B \rangle \mapsto \sum_{x \in \mathcal{W}} \overline{\underline{\mathrm{rk}}}\, \mathrm{Hom}(B, R_x)\, T_x.$$

*Proof.* This theorem is a direct consequence of 5.7 and 5.16, which we will prove in the sequal without using it. □

**Definition 5.4.** For any $R$-bimodule $B$ and any subset $A \subset \mathcal{W}$ we define the subbimodule

$$\Gamma_A B = \{b \in B \mid \operatorname{supp} b \subset \operatorname{Gr}(A)\}$$

of all elements with support in $\operatorname{Gr}(A)$. We put $\Gamma_{\geq i} B = \Gamma_{\{x \in \mathcal{W} | l(x) \geq i\}} B$ and define the category

$$\mathcal{F}_\Delta \subset \mathcal{R}$$

as the full subcategory of all graded bimodules $B \in \mathcal{R}$ such that $B$ has its support in a set of the form $\operatorname{Gr}(A)$ for some finite $A \subset \mathcal{W}$ and tht the quotient $\Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ is for all $i$ isomorphic to a finite direct sum of graded bimodules of the form $R_x[\nu]$ with $l(x) = i$ and $\nu \in \mathbb{Z}$.

*Remark* 5.5. Certainly $B \mapsto \Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ is additive in $B$, thus $\mathcal{F}_\Delta$ is stable under forming direct sums and because of Krull-Schmid 1.3 also stable under forming direct summands.

*Notation* 5.6. It turns out to be convenient to introduce the graded bimodules

$$\Delta_x = R_x[-l(x)]$$

and to work in the Hecke algebra with $\tilde{T}_x = v^{l(x)} T_x$. for the multiplicity of a summand $\Delta_x[\nu]$ in one and every decomposition of $\Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ for $i = l(x)$ we introduce the notation $(B : \Delta_x[\nu])$. Also we introduce the abreviation $R[1] \otimes_{R^s} M = \theta_s M$.

**Proposition 5.7.** *Let $s \in \mathcal{S}$ be a simple reflection.*

1. *With $B$ also $R \otimes_{R^s} B$ belongs to $\mathcal{F}_\Delta$.*
2. *If we define the maps $h_\Delta : \mathcal{F}_\Delta \to \mathcal{H}$ by the rule $B \mapsto \sum_{x,\nu} (B : \Delta_x[\nu]) v^\nu \tilde{T}_x$, for all $s \in \mathcal{S}$ we get two commutative diagrams*

$$\begin{array}{ccc} \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \\ {\scriptstyle \theta_s}\downarrow & & \downarrow{\scriptstyle (\tilde{T}_s + v)\cdot} \\ \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \\ {\scriptstyle [1]}\downarrow & & \downarrow{\scriptstyle v\cdot} \\ \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \end{array}$$

3. *Our map $\mathcal{E}$ factors via a map $\mathcal{E} : \mathcal{H} \to \langle \mathcal{F}_\Delta \rangle$ and our $h_\Delta : \langle \mathcal{F}_\Delta \rangle \to \mathcal{H}$ is left inverse to this $\mathcal{E}$*

*Proof.* The proof of this proposition needs

**Lemma 5.8.** *Let $W$ be a finite dimensional vector space and $U, V \subset W$ two affine subspaces. Then $\mathrm{Ext}^1_{R(W)}(R(U), R(V))$ is nonzero only if we have $V \cap U = V$ or if $V \cap U$ is of codimension one in $V$. In the later case $\mathrm{Ext}^1_{R(W)}(R(U), R(V))$ is a free $R(U \cap V)$-modul of rank one, generated by the class of an arbitray short exact sequence of the form*

$$R(V)[-2] \stackrel{\alpha}{\hookrightarrow} R(U \cup V) \twoheadrightarrow R(U)$$

*for $\alpha \in W^*$ with $\alpha|_U = 0$, $\alpha|_V \neq 0$.*

*Proof.* If $F$ and $G$ are free modules of finite rank over the $k$-algebras $A$ and $B$ and if $M$ or $N$ are arbitray modules over $A$ and $B$, then we obviously (see [Bou70], I §4 and II §11) have

$$\mathrm{Hom}_A(F, M) \otimes \mathrm{Hom}_B(G, N) \stackrel{\sim}{\to} \mathrm{Hom}_{A \otimes B}(F \otimes G, M \otimes N).$$

If our algebras are noetherian and $M'$ resp. $N'$ are finitely generated over $A$ resp. $B$, we find resolutions $F^\bullet \twoheadrightarrow M'$ resp. $G^\bullet \twoheadrightarrow N'$ with $F^i$ resp. $G^j$ free of finite rank over $A$ resp. $B$. Then $F^\bullet \otimes G^\bullet$ will be a free resolution of $M' \otimes N'$ and we deduce

$$\begin{aligned} \mathrm{Ext}^n_{A \otimes B}(M' \otimes N', M \otimes N) &= H^n \,\mathrm{Hom}_{A \otimes B}(F^\bullet \otimes G^\bullet, M \otimes N) \\ &= H^n(\mathrm{Hom}_A(F^\bullet, M) \otimes \mathrm{Hom}_B(G^\bullet, N)) \\ &= \textstyle\bigoplus_{i+j=n} \mathrm{Ext}^i_A(M', M) \otimes \mathrm{Ext}^j_B(N', N) \end{aligned}$$

Thus we can restrict in our Lemma to the case with $\dim_k W = 1$, and those can easily be done in an explicit way. More precisely we may without restriction assume $U$ and $V$ linear and find a decomposition by $W = S \oplus U' \oplus V' \oplus W'$ mit $U = S \oplus U'$ and $V = S \oplus V'$. If we denote the dimension by $\dim W = s + u + v + w$, we will get $\mathrm{Ext}^\bullet_{R(W)}(R(U), R(V)) \cong \mathrm{Ext}^\bullet_{k[X]}(k[X], k[X])^s \otimes \mathrm{Ext}^\bullet_{k[X]}(k[X], k)^u \otimes \mathrm{Ext}^\bullet_{k[X]}(k, k[X])^v \otimes \mathrm{Ext}^\bullet_{k[X]}(k, k)^w$. Thus $\mathrm{Ext}^1$ is possible only for $v \leq 1$ and in case $v = 1$ we get $\mathrm{Ext}^1 = k[X]^s$. □

Now we prove the proposition. Let $s \in \mathcal{S}$ be our fixed simple reflection. Certainly we can refine our filtration $\Gamma_{\geq i}$ of $B$ by certain $\Gamma_{\geq j} B$ with $j \in \mathbb{Z} + 0,5$ such that for all $i \in \mathbb{Z}$ the quotients $\Gamma_{\geq i} B / \Gamma_{\geq i+0,5} B$ resp. $\Gamma_{\geq i-0,5} B / \Gamma_{\geq i} B$ are sums of $R(x)[\nu]$ with $x > sx$ resp. $x < sx$. With these choices the parameters $x, y$ of two possible subquotients of $\Gamma_{\geq i-0,5} B / \Gamma_{\geq i+0,5} B$ only differ by a reflection in case $y = sx$: Indeed for an arbitray reflection $t$ we always have $tx > x$ or $tx < x$, and $sy > y < x > sx$ implies $y \leq sx$ after a result of Deodhar, the so-called property Z of Coxeter groups. However all extensions in $\mathrm{Ext}^1_{R \otimes R}(R_x, R_{sx})$ die after restriction to $R^s \otimes R$. Indeed the restriction $R_{x,sx} \twoheadrightarrow R_x$ splits over $R^s \otimes R$, since in the notiations of 4.5 it gives an isomorphism $R^+_{x,sx} \xrightarrow{\sim} R_x$. By 5.8 thus the restriction onto $R^s \otimes R$ of $\Gamma_{\geq i-0,5}B/\Gamma_{\geq i+0,5}B$ is isomorphic to a direct sum of copies of certain $R_x[\nu]$ with $x > sx$ and $l(x) = i$, and these occur with multiplicity $(B : R_x[\nu]) + (B : R_{sx}[\nu])$. If we now tensor with $R\otimes_{R^s}$ and remember the short exact sequences $R_x[-2] \hookrightarrow R \otimes_{R^s} R_x \twoheadrightarrow R_{sx}$, we get the first claim and (always assuming $x > sx$) the formulas

$$\begin{array}{lcll} (R \otimes_{R^s} B : R_x[\nu]) & = & (B : R_x[\nu + 2]) & + (B : R_{sx}[\nu + 2]) \\ (R \otimes_{R^s} B : R_{sx}[\nu]) & = & (B : R_x[\nu]) & + (B : R_{sx}[\nu]) \end{array}$$

This we can rewrite to

$$\begin{array}{lcll} (\theta_s B : \Delta_x[\nu]) & = & (B : \Delta_x[\nu + 1]) & + (B : \Delta_{sx}[\nu]) \\ (R[1] \otimes_{R^s} B : \Delta_{sx}[\nu]) & = & (B : \Delta_x[\nu]) & + (B : \Delta_{sx}[\nu - 1]) \end{array}$$

If we now only for this proof write an element $H \in \mathcal{H}$ in the form $H = \sum_{x,\nu}(H : v^\nu \tilde{T}_x)\, v^\nu \tilde{T}_x$, then in the Hecke algebra we get simalarily

$$\begin{array}{lcll} ((\tilde{T}_s + v)H : v^\nu \tilde{T}_x) & = & (H : v^{\nu+1}\tilde{T}_x) & + (H : v^\nu \tilde{T}_{sx}) \\ ((\tilde{T}_s + v)H : v^\nu \tilde{T}_{sx}) & = & (H : v^\nu \tilde{T}_x) & + (H : v^{\nu-1}\tilde{T}_{sx}) \end{array}$$

This shows the second claim. To prove the last claim, we Remark first, that we may consider $\langle\mathcal{R}\rangle$ via the ring homomorphism $\mathcal{E}$ of 1.10 also as a left $\mathcal{H}$modul. The first part of the proposition then means, that $\langle\mathcal{F}_\Delta\rangle \subset \langle\mathcal{R}\rangle$ is an $\mathcal{H}$-submodul ist and the second part, that the map $h_\Delta : \langle\mathcal{F}_\Delta\rangle \to \mathcal{H}$ is a homomorphism of $\mathcal{H}$-modules. Since $\mathcal{E}(1) = \langle R_e\rangle$ belongs to $\mathcal{F}_\Delta$, the map $\mathcal{E}$ factorizes over $\langle\mathcal{F}_\Delta\rangle$, and since $h_\Delta \circ \mathcal{E} : \mathcal{H} \to \mathcal{H}$ is a homomorphism of $\mathcal{H}$-modules mapping one to one, this composition has to be the identity. □

To show theorem 5.3 we als need a "dual" approach. More precisely we consider the filtration of our bimodules by the $\Gamma_{\leq i}B = \Gamma_{\{x\in\mathcal{W}|l(x)\leq i\}}B$ and define the category

$$\mathcal{F}_\nabla \subset \mathcal{R}$$

as the full subcategory of all graded bimodules $B \in \mathcal{R}$ such, that the support is in $\mathrm{Gr}(A)$ for some finite $A \subset \mathcal{W}$ and that the quotients are for all $i$ isomorphic to finite direct sum of graded bimodules of the form $R_x[\nu]$ with $l(x) = i$ and $\nu \in \mathbb{Z}$. In this context it is convenient and natural, to introduce

$$\nabla_x = R_x[l(x)].$$

For the multiplicitz of $\nabla_x[\nu]$ in a direct sum decomposition of $\Gamma_{\leq l(x)}B/\Gamma_{\leq l(x)-1}B$ we use the notation $(B : \nabla_x[\nu])$.

Remark that the multiplicities $(B : \nabla_x[\nu])$ and $(B : \Delta_x[\nu])$ concern subquotients for different filtrations. Even for $B \in \mathcal{F}_\Delta \cap \mathcal{F}_\nabla$ the multiplicities $(B : \Delta_x[l(x) + \nu])$ and $(B : \nabla_x[-l(x) + \nu])$ are thus different in general. Analogously we have now

**Proposition 5.9.** *Let $s \in \mathcal{S}$ be a simple reflection.*

1. *With $B$ also $R \otimes_{R^s} B$ belongs to $\mathcal{F}_\nabla$.*
2. *When we define th map $h_\nabla : \mathcal{F}_\nabla \to \mathcal{H}$ by the rule $B \mapsto \sum_{x,\mu}(B : \nabla_x[\mu])\, v^{-\mu}\tilde{T}_x$, then for all $s \in \mathcal{S}$ the following diagrams are commutative*

$$\begin{array}{ccc} \mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \\ {\scriptstyle\theta_s}\downarrow & & \downarrow{\scriptstyle(\tilde{T}_s+v)\cdot} \\ \mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \\ {\scriptstyle[1]}\downarrow & & \downarrow{\scriptstyle v^{-1}\cdot} \\ \mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \end{array}$$

3. *The composition $d \circ h_\nabla$ is a left inverse of $\mathcal{E} : \mathcal{H} \to \langle \mathcal{F}_\nabla \rangle$.*

*Proof.* We consider the functor

$$D = \mathrm{Hom}_{-R}(\ , R) : \mathcal{R} \to R\text{-mod}_{\mathbb{Z}}\text{-} R,$$

where we provide our space of homomorphisms of right $R$-modules with the obvious $\mathbb{Z}$-grading and let the right resp. left action on $DB$ be defined via the right resp. left action on the bimodule $B$, in formulas $(rf)(b) = f(rb)$ and $(fr)(b) = f(br)$ for all $b \in B$, $r \in R$, $f \in DB$. I want to avoid a general discussion whether an object of $\mathcal{R}$ lands in $\mathcal{R}$ again under $D$. In any case we have $DR_x \cong R_x$ and $D(M[\nu]) = (DM)[-\nu]$, our functor induces an equivalence of categories $D : \mathcal{F}_\nabla \xrightarrow{\sim} \mathcal{F}_\Delta^{\mathrm{opp}}$ and evidently we have $h_\nabla = h_\Delta \circ D$. Thus we only need to establish for all $M \in \mathcal{F}_\nabla$ an isomorphism $\theta_s DM \cong D\theta_s M$. For that we first show:

**Proposition 5.10.** 1. *The functors from $R^s$-$\mathrm{mod}_{\mathbb{Z}}$ to $R$-$\mathrm{mod}_{\mathbb{Z}}$ with $M \mapsto R[2] \otimes_{R^s} M$ and $M \mapsto \mathrm{Hom}_{R^s}(R, M)$ are naturally equivalent.*

2. *The functor $R[1] \otimes_{R^s} : R$-$\mathrm{mod}_{\mathbb{Z}} \to R$-$\mathrm{mod}_{\mathbb{Z}}$ is selfadjoint.*

*Proof.* Since $R$ is free of finite rank over $R^s$, our Hom −functor can can also be written in the form

$$\mathrm{Hom}_{R^s}(R, -) = \mathrm{Hom}_{R^s}(R, R^s) \otimes_{R^s} -.$$

Now more precisely $R$ is free over $R^s$ with basis $1, \alpha$ for $\alpha$ an equation of the reflecting hyperplane $s(\alpha) = -\alpha$. The dual basis of $\mathrm{Hom}_{R^s}(R, R^s)$ over $R^s$

will be denoted $1^*, \alpha^*$. The multiplication by $\alpha \in R$ is given in this basis by $\alpha\alpha^* = 1^*$, $\alpha 1^* = \alpha^2\alpha^*$. The choice of $\alpha$ therefore defines an isomorphism of $R$-moduls

$$R[2] \xrightarrow{\sim} \mathrm{Hom}_{R^s}(R, R^s), 1 \mapsto \alpha^*, \alpha \mapsto 1^*,$$

and the first claim is established. The second claim follows, since our two functors from the first claim are up to a change of grading just the two adjoints of the restriction of $R^s$. □

Thus we indeed get isomorphisms

$$\begin{aligned} \theta_s(DM) &\cong R[1] \otimes_{R^s} DM \\ &\cong \mathrm{Hom}_{R^s}(R[1], \mathrm{Hom}_{-R}(M, R)) \\ &\cong \mathrm{Hom}_{-R}(\theta_s M, R) \\ &\cong D(\theta_s M) \end{aligned}$$

The third part of Proposition 5.9 now follows, since we have $d(\tilde{T}_s + v) = \tilde{T}_s + v$ and thus the composition $d \circ h_\nabla \circ \mathcal{E} : \mathcal{H} \to \mathcal{H}$ is a homomorphism of $\mathcal{H}$-leftmodules with $1 \mapsto 1$. □

**Definition 5.11.** We now define the category $\mathcal{B} \subset \mathcal{R}$ as the category of all graded bimodules $B \in \mathcal{R}$, whose class $\langle B \rangle$ lies in the image of our morphism $\mathcal{E} : \mathcal{H} \to \langle \mathcal{R} \rangle$, and call the objects of $\mathcal{B}$ our **special bimodules**.

*Remark* 5.12. Certainly $\mathcal{B}$ is stable under direct sums and shifts of grading. Much later in 6.16 we will be able to show that $\mathcal{B}$ is stable under direct summands. To obtain a criterium for when a bimodul $B$ belongs to $\mathcal{B}$, we consider for an arbitrary finite sequence $\underline{s} = (r, \ldots, t)$ of simple reflections in the Hecke algebra the element $b(\underline{s}) = (T_r + 1) \ldots (T_t + 1)$ and form the bimodule

$$B(\underline{s}) = R \otimes_{R^r} \ldots R \otimes_{R^t} R.$$

**Lemma 5.13.** *A graded bimodule $B \in \mathcal{R}$ is special iff there exist objects $C, D \in \mathcal{R}$, which each are finite direct sums of objects of the form $B(\underline{s})[n]$ and such, that we have*

$$B \oplus C \cong D$$

*Proof.* Since $\langle B(\underline{s})[n] \rangle = \mathcal{E}(v^n b(\underline{s}))$ the $B(\underline{s})[n]$ are special. This shows our criterium to be sufficient. Since the $v^n b(\underline{s})$ generate $\mathcal{H}$ as an abelian group, it is necessary. □

*Remark* 5.14. In particular $\mathcal{E} : \mathcal{H} \to \langle \mathcal{R} \rangle$ factorizes over the split Grothendieck group $\langle \mathcal{B} \rangle$ of the additive category $\mathcal{B}$ and 5.7 with 5.9 imply $\mathcal{B} \subset \mathcal{F}_\Delta \cap \mathcal{F}_\nabla$.

**Theorem 5.15.** *For $M \in \mathcal{F}_\Delta$, $N \in \mathcal{B}$ and also for $M \in \mathcal{B}$, $N \in \mathcal{F}_\nabla$ the space* $\mathrm{Hom}(M,N)$ *is graded free as a right $R$-module with rank*

$$\underline{\mathrm{rk}}\,\mathrm{Hom}(M,N) = \sum_{x,\nu,\mu} (M : \Delta_x[\nu])(N : \nabla_x[\mu]) v^{\mu-\nu}.$$

*Proof.* We only treat the case $M \in \mathcal{F}_\Delta$, $N \in \mathcal{B}$, the other case is similar. Let $i : \mathcal{H} \to \mathcal{H}$ denote the antiinvolution $i(v) = v$, $i(\tilde{T}_x) = \tilde{T}_{x^{-1}}$. We consider the symmetric $\mathbb{Z}[v,v^{-1}]$-bilinear pairing $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{Z}[v,v^{-1}]$, which to each pair $(F,G)$ associates the coefficient of $\tilde{T}_e = T_e$ in the representation of $i(F)G$ as a linear combination of the $T$ or equivalently the $\tilde{T}$. It can be described explicitly by $\langle \tilde{T}_x, \tilde{T}_y \rangle = \delta_{xy}$, see [Lus85],§1.4. The looked-for formula thus gets the form

$$\overline{\underline{\mathrm{rk}}}\,\mathrm{Hom}(M,N) = \langle h_\Delta M, h_\nabla N \rangle.$$

By 5.13 we can restrict to the case $N = B(\underline{s})$. With 5.10 we can see, that our formula is correct for the pair $(R \otimes_{R^s} M, N)$ iff it is correct for the pair $(M, R \otimes_{R^s} N)$. Without any difficulties we also see, that our formula is correct for $(M[1], N)$ if and only if it is correct for $(M, N[-1])$. Using this we can even restrict to the case $N = R_e$ which is obvious. □

**Korollar 5.16.** *Let $B \in \mathcal{B}$ be a special bimodule. Then we have*

$$\begin{aligned} h_\Delta(B) &= \textstyle\sum_{x \in \mathcal{W}} \overline{\underline{\mathrm{rk}}}\,\mathrm{Hom}(B, R_x)\, T_x, \\ h_\nabla(B) &= \textstyle\sum_{x \in \mathcal{W}} \overline{\underline{\mathrm{rk}}}\,\mathrm{Hom}(R_x, B)\, T_x. \end{aligned}$$

*Proof.* By the preceding 5.15 we have

$$\begin{aligned} \underline{\mathrm{rk}}\,\mathrm{Hom}(B, R_x) &= \underline{\mathrm{rk}}\,\mathrm{Hom}(B, \nabla_x[-l(x)]) \\ &= \textstyle\sum_\nu (B : \Delta_x[\nu]) v^{-l(x)-\nu} \\ \underline{\mathrm{rk}}\,\mathrm{Hom}(R_x, B) &= \underline{\mathrm{rk}}\,\mathrm{Hom}(\Delta_x[l(x)], B) \\ &= \textstyle\sum_\mu (B : \nabla_x[\mu]) v^{-l(x)+\mu} \end{aligned}$$

and the corollary follows from the definitions. □

# 6 Classification of indecomposable special bimodules

*Notation* 6.1. We work with a reflection faithful representation over an infinite field. For $B \in \mathcal{B}$ and $y \in \mathcal{W}$ we abbreviate $\Gamma_{\leq y} B / \Gamma_{<y} B = \Gamma_y^{\leq} B$, and also $\Gamma_{\geq y} B / \Gamma_{>y} B = \Gamma_y^{\geq} B$ and $B / \Gamma_{\neq y} B = \Gamma^y B$.