THE HODGE THEORY OF SOERGEL BIMODULES

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Abstract. We prove Soergel's conjecture on the characters of indecomposable Soergel bimodules. We deduce that Kazhdan-Lusztig polynomials have positive coefficients for arbitrary Coxeter systems. Using results of Soergel one may deduce an algebraic proof of the Kazhdan-Lusztig conjecture.

Contents

1. Introduction

In 1979 Kazhdan and Lusztig introduced the Kazhdan-Lusztig basis of the Hecke algebra of a Coxeter system [KL1]. The definition of the Kazhdan-Lusztig basis is elementary, however it appears to enjoy remarkable positivity properties. For example, it is conjectured in [KL1] that Kazhdan-Lusztig polynomials (which express the Kazhdan-Lusztig basis in terms of the standard basis of the Hecke algebra) have positive coefficients. The same paper also proposed the Kazhdan-Lusztig conjecture, a character formula for simple highest weight modules for a complex semi-simple Lie algebra in terms of Kazhdan-Lusztig polynomials associated to its Weyl group.

In a sequel [KL2], Kazhdan and Lusztig established that their polynomials give the Poincaré polynomials of the local intersection cohomology of Schubert varieties (using Deligne's theory of weights), thus establishing their positivity conjectures for finite and affine Weyl groups. In 1981 Beilinson and Bernstein [BB] and Brylinski and Kashiwara [BK] established a connection between highest weight representation theory and perverse sheaves, using D-modules and the Riemann-Hilbert correspondence, thus proving the Kazhdan-Lusztig conjecture. Since their introduction Kazhdan-Lusztig polynomials have become ubiquitous throughout highest weight representation theory, giving character formulae for affine Lie algebras, quantum groups at a root of unity, rational representations of algebraic groups, etc.

In 1990 Soergel [S1] gave an alternate proof of the Kazhdan-Lusztig conjecture, using certain modules over the cohomology ring of the flag variety.[1] In a subsequent paper [S2] Soergel introduced equivariant analogues of these modules, which have come to be known as *Soergel bimodules*.

Soergel's approach is remarkable in its simplicity. Using only the action of the Weyl group on a Cartan subalgebra, Soergel associates to each simple reflection a graded bimodule over the regular functions on the Cartan subalgebra. He then proves that the split Grothendieck group of the monoidal category generated by these bimodules (the category of Soergel bimodules) is isomorphic to the Hecke algebra. Moreover, the Kazhdan-Lusztig conjectures (as well as several positivity conjectures) are equivalent to the existence of certain bimodules whose classes in the Grothendieck group coincide with the Kazhdan-Lusztig basis. Despite its elementary appearance, this statement is difficult to verify. For finite Weyl groups, Soergel deduces the existence of such bimodules by applying the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber [BBD] to identify the indecomposable Soergel bimodules with the equivariant intersection cohomology of Schubert varieties. This approach was extended by Härterich to the setting of Weyl groups of symmetrizable Kac-Moody groups [Hä]. Except for his appeal to the decomposition theorem, Soergel's approach is entirely algebraic. (The decomposition theorem relies on the base field having characteristic 0, which will be an important assumption below.)

[1] [S1, §1.1, Bermerkung 5]. This seems not to be as well-known as it should be.

In [S2] and [S4] Soergel pointed out that the algebraic theory of Soergel bimodules can be developed for an arbitrary Coxeter system. Starting with an appropriate representation of the Coxeter group (the substitute for the Weyl group's action on a Cartan subalgebra) one defines the monoidal category of Soergel bimodules by mimicking the Weyl group case. Surprisingly, one again obtains a monoidal category whose split Grothendieck group is canonically identified with the Hecke algebra. Soergel then conjectures the existence (over a field of characteristic 0) of indecomposable bimodules whose classes coincide with the Kazhdan-Lusztig basis of the Hecke algebra. At this level of generality there is no known recourse to geometry. One does not have a flag variety or Schubert varieties associated to arbitrary Coxeter groups, and so one has no geometric setting in which to apply the decomposition theorem. Soergel's conjecture was established for dihedral groups by Soergel [S2] and for "universal" Coxeter systems (where each product of simple reflections has infinite order) by Fiebig [F2] and Libedinsky. However, in both these cases there already existed closed formulas for the Kazhdan-Lusztig polynomials.

In this paper we prove Soergel's conjecture for an arbitrary Coxeter system. We thus obtain a proof of the positivity of Kazhdan-Lusztig polynomials (as well as several other positivity conjectures). We also obtain an algebraic proof of the Kazhdan-Lusztig conjecture, completing the program initiated by Soergel. In some sense we have come full circle: the original paper of Kazhdan and Lusztig was stated in the generality of an arbitrary Coxeter system, this paper returns Kazhdan-Lusztig theory to this level of generality.

Our proof is inspired by two papers of de Cataldo and Migliorini ([dCM1] and [dCM2]) which give Hodge-theoretic proofs of the decomposition theorem. In essence, de Cataldo and Migliorini show that the decomposition theorem for a proper map (from a smooth space) is implied by certain Hodge theoretic properties of the cohomology groups of the source, under a Lefschetz operator induced from the target. We discuss their approach in more detail below. Thus they are able to transform a geometric question on the target into an algebraic question on the source. They then use classical Hodge theory and some ingenious arguments to complete the proof. For Weyl groups, Soergel bimodules are the equivariant intersection cohomology of Schubert varieties, and as such have a number of remarkable Hodge-theoretic properties which seem not to have been made explicit before. In fact, these properties hold for any Coxeter group; Soergel bimodules always behave as though they were intersection cohomology spaces of projective varieties! In this paper, we give an algebraic proof of these Hodge-theoretic properties, for any Coxeter group, and adapt the proof that these Hodge-theoretic properties imply the "decomposition theorem", at least insofar as Soergel's conjecture is concerned.

Here are some highlights of de Cataldo and Migliorini's proof from [dCM1]:

(1) "Local intersection forms" (which control the decomposition of the direct image of the constant sheaf) can be embedded into "global intersection forms" on the cohomology of smooth varieties.
(2) The Hodge-Riemann bilinear relations can be used to conclude that the restriction of a form to a subspace (i.e. the image of a local intersection form) stays definite.
(3) One should first prove the hard Lefschetz theorem, and then deduce the Hodge-Riemann bilinear relations via a limiting argument from a family of known cases, using that the signature of a non-degenerate symmetric real form cannot change in a family.

It is this outline that we adapt to our algebraic situation. However the translation of their results into the language of Soergel bimodules is by no means automatic. The biggest obstacle is to find a replacement for the use of hyperplane sections and the weak Lefschetz theorem. We believe that our use of the Rouquier complex to overcome this difficulty is an important observation and may have other applications.

There already exists a formidable collection of algebraic machinery, developed by Soergel [S2, S5], Andersen-Jantzen-Soergel [AJS], and Fiebig [F1, F3], which provides algebraic proofs of many deep results in representation theory once Soergel's conjecture is known. These include the Kazhdan-Lusztig conjecture for affine Lie algebras (in non-critical level), the Kazhdan-Lusztig conjectures for quantum groups at a root of unity, and the Lusztig conjecture on modular characters of reductive algebraic groups in characteristic $p \gg 0$.

There are many formal similarities between the theory we develop here, and the theory of intersection cohomology of non-rational polytopes, which was developed to prove Stanley's conjecture on the unimodularity of the generalized h-vector [BL, Ka, BBFK]. In both cases one obtains spaces which look like the intersection cohomology of a (in many cases non-existent) projective algebraic variety. Dyer [D1, D2] has a proposed a conjectural framework for understanding both of these theories in parallel. It would be interesting to know whether the techniques of this paper shed light on this more general theory.

1.1. **Results.** Fix a Coxeter system (W, S). Let $\mathcal{H}$ denote the Hecke algebra of (W, S), with standard basis $\{H_x\}_{x \in W}$ and Kazhdan-Lusztig basis $\{\underline{H}_x\}_{x \in W}$. We fix a reflection faithful (in the sense of [S4, Definition 1.5]) representation $\mathfrak{h}$ of W over $\mathbb{R}$ and let R denote the regular functions on $\mathfrak{h}$, graded with $\deg \mathfrak{h}^* = 2$. We denote by $\mathcal{B}$ the category of Soergel bimodules; it is the full additive monoidal Karoubian subcategory of graded R-bimodules generated by $B_s := R \otimes_{R^s} R(1)$ for all $s \in S$ (here $R^s \subset R$ denotes the subalgebra of s-invariants). For any x there exists up to isomorphism a unique indecomposable Soergel bimodule B_x which occurs as a direct summand of the Bott-Samelson bimodule $BS(\underline{x}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u$ for any reduced expression $\underline{x} = st \dots u$ for x, but does not occur as a summand of any Bott-Samelson bimodule for a shorter expression. The bimodules B_x for $x \in W$ give representatives for the isomorphism classes of all indecomposable Soergel bimodules up to shifts. The split Grothendieck group $[\mathcal{B}]$ of the category of Soergel bimodules is isomorphic to $\mathcal{H}$. The character $\mathrm{ch}(B) \in \mathcal{H}$ of a Soergel bimodule B is an $\mathbb{Z}_{\geq 0}[v^{\pm}]$-linear combination of standard basis elements $\{H_x\}$ given by counting ranks of subquotients in a certain canonical filtration; it realizes the class of B under the isomorphism $[\mathcal{B}] \xrightarrow{\sim} \mathcal{H}$.

Theorem 1.1. *(Soergel's conjecture) For all $x \in W$ we have* $\mathrm{ch}(B_x) = \underline{H}_x$.

Because $\mathrm{ch}(B)$ is manifestly positive we obtain:

Corollary 1.2. *(Kazhdan-Lusztig positivity conjecture)*

(1) *If we write $\underline{H}_x = \sum_{y \leq x} h_{y,x} H_y$ then $h_{y,x} \in \mathbb{Z}_{\geq 0}[v]$.*
(2) *If we write $\underline{H}_x \underline{H}_y = \sum \mu^z_{x,y} \underline{H}_z$ then $\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.*

We prove that Soergel bimodules have all of the algebraic properties known for intersection cohomology. Given a Soergel bimodule B, we denote by $\overline{B} := B \otimes_R \mathbb{R}$ the quotient by the image of positive degree polynomials acting on the right. We let $(\overline{B})^i$ denote the degree i component of $\overline{B}$. The self-duality of Soergel bimodules implies that $\dim_{\mathbb{R}}(\overline{B_x})^{-i} = \dim_{\mathbb{R}}(\overline{B_x})^i$ for all i. For the rest of the introduction let us fix a degree two element $\rho \in \mathfrak{h}^*$ which is strictly positive on any simple coroot $\alpha_s^\vee \in \mathfrak{h}$ (see § 3.1).

Theorem 1.3. *(Hard Lefschetz for Soergel bimodules) The action of ρ on B_x by left multiplication induces an operator on $\overline{B_x}$ which satisfies the hard Lefschetz theorem. That is, left multiplication by ρ^i induces an isomorphism*

$$\rho^i : (\overline{B_x})^{-i} \xrightarrow{\sim} (\overline{B_x})^i.$$

We say that a graded R-valued form

$$\langle -, - \rangle : B_x \times B_x \to R$$

is *invariant* if it is bilinear for the right action of R, and if $\langle rb, b' \rangle = \langle b, rb' \rangle$ for all $b, b' \in B$ and $r \in R$. Theorem 1.1 and Soergel's hom formula imply that the degree zero endomorphisms of B_x consist only of scalars, i.e. $\mathrm{End}(B_x) = \mathbb{R}$. Combining this with the self-duality of indecomposable Soergel bimodules, we see that there exists an invariant form $\langle -, - \rangle_{B_x}$ on B_x which is unique up to a scalar. We write $\langle -, - \rangle_{\overline{B_x}}$ for the $\mathbb{R}$-valued form on $\overline{B_x}$ induced by $\langle -, - \rangle_{B_x}$. We fix the sign on $\langle -, - \rangle_{B_x}$ by requiring that $\langle \overline{c}, \rho^{\ell(x)} \overline{c} \rangle_{\overline{B_x}} > 0$, where c is any generator of $B_x^{-\ell(x)} \cong \mathbb{R}$. With this additional positivity constraint, we call $\langle -, - \rangle_{B_x}$ the *intersection form* on B_x. It is well-defined up to positive scalar.

Theorem 1.4. *(Hodge-Riemann bilinear relations) For all i the Lefschetz form on $(\overline{B_x})^{-i}$ defined by*

$$(\alpha, \beta)^\rho_{-i} := \langle \alpha, \rho^i \beta \rangle_{\overline{B_x}}$$

is $(-1)^{(-\ell(x)+i)/2}$*-definite when restricted to the primitive subspace*

$$P_\rho^{-i} = \ker(\rho^{i+1}) \subset (\overline{B_x})^{-i}.$$

Note that $B_x^{-i} = 0$ unless i and $\ell(x)$ are congruent modulo 2. Throughout this paper we adopt the convention that if m is odd then a space is $(-1)^{\frac{m}{2}}$-definite if and only it is zero. The reader need not worry too much about the sign in this and other Hodge-Riemann statements. Throughout the introduction the form on the lowest non-zero degree will be positive definite, and the signs on primitive subspaces will alternate from there upwards.

As an example of our results, consider the case when W is finite. If $w_0 \in W$ denotes the longest element of W, then $B_{w_0} = R \otimes_{R^W} R(\ell(w_0))$, where R^W denotes the subalgebra of W-invariants in R. Hence

$$\overline{B_{w_0}} = (R \otimes_{R^W} R) \otimes_R \mathbb{R}(\ell(w_0)) = R/((R^W)^+)(\ell(w_0))$$

is the coinvariant ring, shifted so as to have Betti numbers symmetric about zero (here $((R^W)^+)$ denotes the ideal of R generated by elements of R^W of positive degree). The coinvariant ring is equipped with a canonical symmetric form and Theorems 1.3 and 1.4 yield that left multiplication by any ρ in the interior of the dominant chamber of $\mathfrak{h}^*$ satisfies the hard Lefschetz theorem and Hodge-Riemann bilinear relations.

If W is a Weyl group of a compact Lie group G, then the coinvariant ring above is isomorphic to the real cohomology ring of the flag variety of G and the hard Lefschtetz theorem and Hodge-Riemann bilinear relations follow from classical Hodge theory, because the flag variety is a projective algebraic variety. On the other hand if W is not a Weyl group (e.g. a non-crystallographic dihedral group, or a group of type H_3 or H_4) then there is no obvious geometric reason why the hard Lefschetz theorem or Hodge-Riemann bilinear relations should hold. The hard Lefschetz property for coinvariant rings has been studied by a number of authors [MNW, MW, McD] but even for the coinvariant rings of H_3 and H_4 the fact that the Hodge-Riemann bilinear relations hold seems to be new.

1.2. **Outline of the proof.** Our proof is by induction on the Bruhat order, and the hard Lefschetz property and Hodge-Riemann bilinear relations play an essential role along the way. Throughout this paper we employ the following abbreviations for any $x \in W$:

$S(x)$: Soergel's conjecture holds for B_x: Theorem 1.1 holds for x.

$hL(x)$: hard Lefschetz holds for $\overline{B_x}$: Theorem 1.3 holds for x.

$HR(x)$: the Hodge-Riemann bilinear relations hold for $\overline{B_x}$: Theorem 1.4 holds for x.

The abbreviation $hL(< x)$ means that $hL(y)$ holds for all $y < x$. Similar interpretations hold for abbreviations like $S(\leq x)$, etc.

In the statement of the Hodge-Riemann bilinear relations for $\overline{B_x}$, we assume $S(x)$ in order for the intersection form $\langle -, - \rangle_{B_x}$ on B_x to be well-defined up to a positive scalar. However, we need not assume $S(x)$ in order to ask whether a given form on B_x (not necessarily the intersection form) induces a form on $\overline{B_x}$ satisfying the Hodge-Riemann bilinear relations. Note that B_x appears as a summand of the Bott-Samelson bimodule $BS(\underline{x})$ for any reduced expression $\underline{x}$ for x. Bott-Samelson bimodules are equipped with an explicit symmetric non-degenerate *intersection form* defined using the ring structure and a trace on $BS(\underline{x})$ (just as the intersection form on the cohomology of a smooth projective variety is given by evaluating the fundamental class on a product). The following stronger version of $HR(x)$ is more useful in induction steps, as it can be posed without assuming $S(x)$:

$$HR(\underline{x}) : \quad \begin{array}{c} \text{for a fixed embedding } B_x \subset BS(\underline{x}) \\ \text{the Hodge-Riemann bilinear relations hold:} \\ \text{the conclusions of Theorem 1.4 hold for the} \\ \text{restriction of the intersection form on } BS(\underline{x}) \text{ to } B_x. \end{array}$$

(Here and elsewhere an "embedding" of Soergel bimodules means an "embedding as a direct summand".) Together, $S(x)$ and $HR(\underline{x})$ imply that the restriction of the intersection form on $BS(\underline{x})$ to B_x agrees with the intersection form on B_x up to a positive scalar, for any choice of embedding (see Lemma 3.11). In other words:

$$\text{If } S(x) \text{ holds, then } HR(x) \text{ and } HR(\underline{x}) \text{ are equivalent.} \tag{1.1}$$

We now give an outline of the proof. Fix $x \in W$ and $s \in S$ with $xs > x$ and assume $S(< xs)$. By Soergel's hom formula (see Theorem 3.6) this is equivalent to assuming that $\operatorname{End}(B_y) = \mathbb{R}$ for all $y < xs$. Consider the form given by composition

$$(-,-)_y^{x,s} : \operatorname{Hom}(B_y, B_xB_s) \times \operatorname{Hom}(B_xB_s, B_y) \to \operatorname{End}(B_y) = \mathbb{R}.$$

Soergel's hom formula gives an expression for the dimension of these hom spaces in terms of an inner product on the Hecke algebra. Applying this formula one sees that $S(xs)$ is equivalent to the non-degeneracy of this form for all $y < xs$ (see [S4, Lemma 7.1(2)]). Now B_y and B_xB_s are naturally equipped with symmetric invariant bilinear forms and hence there is a canonical identification ("take adjoints")

$$\operatorname{Hom}(B_y, B_xB_s) = \operatorname{Hom}(B_xB_s, B_y).$$

Hence we can view $(-,-)_y^{x,s}$ as a bilinear form on the real vector space $\operatorname{Hom}(B_y, B_xB_s)$. We call this form the *local intersection form*. We consider "Soergel's conjecture with signs":

$$S_\pm(y,x,s) : \text{The form } (-,-)_y^{x,s} \text{ is } (-1)^{(\ell(x)+1-\ell(y))/2}\text{-definite.}$$

This is a priori stronger than Soergel's conjecture. By the above discussion:

$$S(< xs) \text{ and } S_\pm(< xs, x, s) \text{ imply } S(xs). \tag{1.2}$$

In order to prove $S_\pm(y,x,s)$, we must digress and discuss hard Lefschetz and the Hodge-Riemann bilinear relations for $\overline{B_xB_s}$. The connection is explained by (1.3) below. Recall that we have fixed a degree two element $\rho \in R$ such that $\rho(\alpha_s^\vee) > 0$ for all simple coroots $\alpha_s^\vee$. Consider the "hard Lefschetz" condition:

$$hL(x,s): \quad \rho^i : (\overline{B_xB_s})^{-i} \to (\overline{B_xB_s})^i \text{ is an isomorphism.}$$

Because B_{xs} is a direct summand of B_xB_s, $hL(x,s)$ implies $hL(xs)$. They are equivalent if we know $hL(< xs)$, since every other indecomposable summand of B_xB_s is of the form B_y for $y < xs$.

If we fix a reduced expression $\underline{x}$ for x and an embedding $B_x \subset BS(\underline{x})$ then B_x inherits an invariant form from $BS(\underline{x})$ as discussed above. Similarly, B_xB_s is a summand of $BS(\underline{x}s)$ and inherits an invariant form, which we denote $\langle -,-\rangle_{B_xB_s}$. We formulate the Hodge-Riemann bilinear relations for $\overline{B_xB_s}$ as follows:

$$HR(\underline{x},s): \quad \begin{array}{c} \text{the Lefschetz form } (\alpha,\beta)_\rho^{-i} := \langle \alpha, \rho^i\beta\rangle_{\overline{B_xB_s}} \text{ is} \\ (-1)^{(\ell(x)+1-i)/2}\text{-definite on the primitive subspace} \\ P_\rho^{-i} := \ker(\rho^{i+1}) \subset (\overline{B_xB_s})^{-i}. \end{array}$$

Once again, using that $B_xB_s \cong B_{xs} \oplus \bigoplus B_y^{\oplus m_y}$ for some $m_y \in \mathbb{Z}_{\geq 0}$ one may deduce easily that $HR(\underline{x},s)$ implies $HR(\underline{x}s)$ (see Lemma 2.2). However $HR(\underline{x},s)$ is slightly stronger than assuming $HR(\underline{x}s)$ and $HR(y)$ for all $y < xs$ with $m_y \neq 0$, because it fixes the sign of the restricted form. Indeed, $HR(\underline{x},s)$ is equivalent to the statement that the restriction of $\langle -,-\rangle_{B_xB_s}$ to any summand B_y of B_xB_s is $(-1)^{(\ell(xs)-\ell(y))/2}$ times a positive multiple of the intersection form on B_y. If we write $HR(x,s)$, we refer to $HR(\underline{x},s)$ for some unspecified choice of embedding $B_x \subset BS(\underline{x})$.

Recall that the space $\mathrm{Hom}(B_y, B_xB_s)$ is equipped with the local intersection form $(-,-)_y^{x,s}$ and that $(\overline{B_xB_s})^{-\ell(y)}$ is equipped with the Lefschetz form $(-,-)_\rho^{-\ell(y)}$. The motivation for introducing the condition $HR(\underline{x},s)$ is the following (see Theorem 4.1): for any ρ as above there exists an embedding:

$$i : \mathrm{Hom}(B_y, B_xB_s) \hookrightarrow P_\rho^{-\ell(y)} \subset (\overline{B_xB_s})^{-\ell(y)}.$$

Moreover, this embedding is an isometry up to a positive scalar.

Because the restriction of a definite form to a subspace is definite, we obtain:

$$S(<xs) \text{ and } HR(\underline{x},s) \text{ imply } S_\pm(< xs, x, s). \tag{1.3}$$

Combining (1.3) and (1.2) and the above discussion, we arrive at the core statement of our induction:

$$S(<xs) \text{ and } HR(\underline{x},s) \text{ imply } S(\leq xs) \text{ and } HR(\underline{x}s). \tag{1.4}$$

It remains to show that $S(\leq x)$ and $HR(\leq x)$ implies $HR(\underline{x},s)$. This reduces Soergel's conjecture to a statement about the modules $\overline{B_xB_s}$ and their intersection forms.

The reader might have noticed that hL seems to have disappeared from the picture. Indeed, HR is stronger than hL, and one might ask why we wish to treat hL separately. The reason is that it seems extremely difficult to attack $HR(\underline{x}, s)$ directly. As we noted earlier, de Cataldo and Migliorini's method of proving HR consists in proving hL first for a family of operators, and using a limiting argument to deduce HR.

We adapt their limiting argument as follows. For any $\zeta \geq 0$, consider the Lefschetz operator

$$L_\zeta := (\rho \cdot -)\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho \cdot -)$$

which we view as an endomorphism of B_xB_s. Here $(\rho \cdot -)$ (resp. $(\zeta\rho \cdot -)$) denotes the operator of left multiplication on B_x (resp. B_s) by ρ (resp. $\zeta\rho$) and juxtaposition denotes tensor product of operators. Now consider the following "ζ-deformations" of the above statements:

$$hL(x,s)_\zeta : \quad L_\zeta^i : (\overline{B_xB_s})^{-i} \to (\overline{B_xB_s})^{i} \text{ is an isomorphism.}$$

$$HR(\underline{x},s)_\zeta : \quad \begin{array}{c} \text{the Lefschetz form } (\alpha,\beta)^\rho_{-i} := \langle \alpha, L^i_\zeta \beta\rangle_{\overline{B_xB_s}} \text{ is} \\ (-1)^{(\ell(x)+1-i)/2}\text{-definite on the primitive subspace} \\ P^{-i}_{L_\zeta} := \ker(L^i_\zeta) \subset (\overline{B_xB_s})^{-i}. \end{array}$$

Note that L_0 is simply left multiplication by ρ, and hence $hL(x,s)_0 = hL(x,s)$ and $HR(\underline{x},s)_0 = HR(\underline{x},s)$. The signature of a family of non-degenerate symmetric real forms can not change within the family. Therefore, if $hL(x,s)_\zeta$ holds for all $\zeta \geq 0$ and $HR(\underline{x},s)_\zeta$ holds for any single non-negative value of ζ, then $HR(\underline{x},s)_0$ also holds. (This is the essence of de Cataldo and Migliorini's limiting argument.)

The first hint that this deformation is promising is Theorem 5.1:

$$HR(\underline{z}) \text{ implies } HR(\underline{z},s)_\zeta \text{ for } \zeta \gg 0 \tag{1.5}$$

(which holds regardless of whether $zs > z$ or $zs < z$). Therefore, we have

$$hL(x,s)_\zeta \text{ for all } \zeta \geq 0, \text{ implies } HR(\underline{x},s)_\zeta \text{ for all } \zeta \geq 0. \tag{1.6}$$

In particular, the fact that $hL(z,s)_\zeta$ and $HR(\underline{z},s)_\zeta$ hold for all $\zeta \geq 0$ and all $z < x$ with $sz > z$ is something we may inductively assume, when trying to prove the same facts for x.

We have reduced our problem to establishing $hL(x,s)_\zeta$ for $\zeta \geq 0$. In de Cataldo and Migliorini's approach this is established using the weak Lefschetz theorem and the Hodge-Riemann bilinear relations in smaller dimension. In our setting the weak Lefschetz theorem is missing, and a key point is the use of Rouquier complexes as a replacement (see the first few paragraphs of §6 for more details). The usual proof of hL for a vector space V is to find a map $V \to W$ of degree 1, injective on V^{-i} for $i > 0$ and commuting with the Lefschetz operator, where HR is known to hold for W. The Rouquier complex yields a map of degree 1 from B_xB_s, injective on negative degrees and commuting with L, to a direct sum of B_x and terms of the form B_zB_s for summands B_z of $BS(\underline{x})$ with $z < x$. This target space does not satisfy the Hodge-Riemann bilinear relations, but nevertheless we are able to prove the hard Lefschetz theorem.

When $\zeta = 0$, we have an argument which shows:

(1.7)

$S(\leq x)$, $hL(<xs)$ and $HR(z,s)$ for all $z < x$ with $zs > z$ together imply $hL(x,s)$.

This is Theorem 6.23. One feature of the proof is that, whenever $zs < z$, the decomposition $B_z B_s \cong B_z(1) \oplus B_z(-1)$ commutes with the Lefschetz operator L_0. This decomposition allows one to bypass the fact that $HR(z,s)$ fails if $zs < z$.

When $\zeta > 0$, the decomposition $B_z B_s \cong B_z(1) \oplus B_z(-1)$ for $zs < z$ does not commute with L_ζ. However, proving $hL(z,s)_\zeta$ for $\zeta > 0$ and $zs < z$ using $hL(z)$ is a straightforward computation (Theorem 6.21). Our inductive hypotheses and the limiting argument above now yield $HR(\underline{z},s)_\zeta$ for all $z < x$. A similar argument to the previous case shows:

(1.8) For $\zeta > 0$, $S(\leq x)$, $HR(\leq x)$, and $HR(< x,s)_\zeta$ imply $hL(x,s)_\zeta$.

This is Theorem 6.22.

Let us summarize the overall inductive proof. Let $X \subset W$ be an ideal in the Bruhat order (i.e. $z \leq x \in X \implies z \in X$) and assume:

(1) $HR(z,t)_\zeta$ for all $\zeta \geq 0$, $z < zt \in X$ and $t \in S$,
(2) $HR(z,t)_\zeta$ for all $\zeta > 0$, $zt < z \in X$ and $t \in S$.

We have already explained why (1) implies $S(X)$, $hL(X)$ and $HR(X)$.

Now choose a minimal element x' in the complement of X, and choose $s \in S$ and $x \in X$ with $x' = xs$. As we just discussed, (1.7) and (1.8) imply that $hL(x,s)_\zeta$ holds for all $\zeta \geq 0$. Using $HR(\underline{x})$ and (1.5) we deduce $HR(x,s)_\zeta$ for all $\zeta \geq 0$. Therefore (1) holds with X replaced by $X \cup \{x'\}$, and thus $S(x')$, $hL(x')$, and $HR(x')$ all hold.

As above, the straightforward calculations of Theorem 6.21 show that $hL(x',t)_\zeta$ holds for $\zeta > 0$ when $t \in S$ satisfies $x't < x'$. Again by $HR(x')$ and (1.5) we have $HR(x',t)_\zeta$ for all $\zeta > 0$ in this case. Thus (2) holds for $X \cup \{x'\}$ as well.

By inspection, (1) and (2) hold for the set $X = \{w \in W \mid \ell(w) \leq 2\}$. Hence by induction we obtain (1) and (2) for $X = W$. We have already explained why this implies all of the theorems in §1.1.

1.3. **Note to the reader.** In order to keep this paper short and have it cite only available sources, we have written it in the language of [S4]. However [S4] is not an easy paper, and we make heavy use of its results. We did not discover the results of this paper in this language, but rather in the diagrammatic language of [EW1] and [EW2]. These papers also provide simpler proofs of the requisite results from [S4]. Another more leisurely account of our arguments will be given in [EW3], where we also (hope to) treat the Hodge theory of Bott-Samelson bimodules in the spirit of [dCM2].

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The second author dedicates his work to the memory of Leigh, who would not have given two hoots if certain polynomials have positive coefficients!

2. Lefschetz linear algebra

Let $H = \bigoplus_{i\in\mathbb{Z}} H^i$ be a graded finite dimensional real vector space equipped with a non-degenerate symmetric bilinear form

$$\langle -,-\rangle_H : H \otimes_{\mathbb{R}} H \to \mathbb{R}$$

which is *graded* in the sense that $\langle H^i, H^j\rangle = 0$ unless $i = -j$.

Let $L\colon H^\bullet \to H^{\bullet+2}$ denote an operator of degree 2. We may also write $L \in \mathrm{Hom}(H, H(2))$, where (2) indicates a grading shift. We say that L is a *Lefschetz operator* if $\langle Lh, h'\rangle = \langle h, Lh'\rangle$ for all $h, h' \in H$. We assume from now on that L is a Lefschetz operator. We say that L *satisfies the hard Lefschetz theorem* if $L^i : H^{-i} \to H^i$ is an isomorphism for all $i \in \mathbb{Z}_{\geq 0}$. For $i \geq 0$ set

$$P_L^{-i} := \ker L^{i+1} \subset H^{-i}.$$

We call P_L^{-i} the *primitive subspace* of H^{-i} (with respect to L). If L satisfies the hard Lefschetz theorem then we have a decomposition

$$H = \bigoplus_{\substack{i\geq 0\\ 0\leq j\leq i}} L^j P_L^{-i}.$$

This is the *primitive decomposition* of H.

For each $i \geq 0$ we define the *Lefschetz form* on H^{-i} via

$$(h, h')_L^{-i} := \langle h, L^i h'\rangle.$$

All Lefschetz forms are non-degenerate if and only if L satisfies the hard Lefschetz theorem, because $\langle -,-\rangle$ is non-degenerate by assumption. Because L is a Lefschetz operator we have $(h,h')_L^{-i} = (Lh, Lh')_L^{-i+2}$ for all $i \geq 2$ and $h, h' \in H^{-i}$. If L satisfies the hard Lefschetz theorem then the primitive decomposition is orthogonal with respect to the Lefschetz forms.

We say that H is *odd* (resp. *even*) if $H^{\mathrm{even}} = 0$ (resp. $H^{\mathrm{odd}} = 0$). Recall that a bilinear form $(-,-)$ on a real vector space is said to be $+1$ *definite* (resp. -1 *definite*) if (v, v) is strictly positive (resp. negative) for all non-zero vectors v.