Note

Well, Papa, Can You Multiply Triplets?

SOPHIE MORIER-GENOUD AND VALENTIN OVSIENKO

We show that the classical algebra of quaternions is a commutative $\mathbb{Z}_2\times \mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra. A similar interpretation of the algebra of octonions is impossible.

This note is our "private investigation" of what really happened on the 16th of October, 1843 on the Brougham Bridge when Sir William Rowan Hamilton engraved on a stone his fundamental relations:

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1.$$

Since then, the elements i, j and k, together with the unit, 1, have denoted the canonical basis of the celebrated four-dimensional associative algebra of quaternions $\mathbb{H}$.

Of course, the algebra $\mathbb{H}$ is not commutative: The relations above imply that the elements i, j, k *anticommute* with each other, for instance

$$i \cdot j = -j \cdot i = k.$$

Yes, but...

The Algebra of Quaternions Is a Graded Commutative Algebra

Our starting point is the following observation.

> *The algebra $\mathbb{H}$ indeed satisfies a graded commutativity condition.*

Let us introduce the following "triple degree":

$$\begin{aligned} \sigma(1) &= (0,0,0),\\ \sigma(i) &= (0,1,1),\\ \sigma(j) &= (1,0,1),\\ \sigma(k) &= (1,1,0). \end{aligned} \tag{1}$$

Then, quite remarkably, the usual product of quaternions satisfies the graded commutativity condition:

$$p \cdot q = (-1)^{\langle \sigma(p), \sigma(q)\rangle}\, q \cdot p, \tag{2}$$

provided each of $p, q \in \mathbb{H}$ is proportional to one of the basis vectors and where $\langle\ ,\ \rangle$ is the scalar product of 3-vectors. Indeed, $\langle\sigma(i),\ \sigma(j)\rangle = 1$ and similarly for k, so that i, j and k anticommute with each other, but $\langle\sigma(i),\ \sigma(i)\rangle = 2$. The product $i \cdot i$ of i with itself is commutative and similarly for j and k, without any contradiction.

The degree (1) viewed as an element of the abelian group $\mathbb{Z}_2\times \mathbb{Z}_2\times\mathbb{Z}_2$ satisfies the following linearity condition

$$\sigma(x\cdot y) = \sigma(x) + \sigma(y), \tag{3}$$

for all homogeneous $x, y \in \mathbb{H}$. The relations (2) and (3) together mean that $\mathbb{H}$ is a $\mathbb{Z}_2\times \mathbb{Z}_2\times\mathbb{Z}_2$ - graded commutative algebra.

We did not find the above observation in the literature (see however [1] for a different "abelianization" of $\mathbb{H}$ in terms of a twisted $\mathbb{Z}_2\times \mathbb{Z}_2$ group algebra; see also [2, 3, 4]). Its main consequence is a systematic procedure of *quaternionization* (similar to complexification). Indeed, many classes of algebras allow tensor product with commutative algebras. Let us give an example. Given an arbitrary real Lie algebra $\mathfrak{g}$, the tensor product $\mathfrak{g}_{\mathbb{H}} := \mathbb{H}\otimes_{\mathbb{R}}\mathfrak{g}$ is a $\mathbb{Z}_2\times \mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie algebra. If furthermore $\mathfrak{g}$ is a real form of a simple complex Lie algebra, then $\mathfrak{g}_{\mathbb{H}}$ is again simple.

The above observation gives a general idea of studying graded commutative algebras over the abelian group

$$\Gamma = \underbrace{\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2}_{n\,\text{times}}.$$

One can show that, in some sense, this is the most general grading, in the graded-commutative-algebra context, but we will not provide the details here. Let us mention that graded commutative algebras are essentially studied in the case $\Gamma = \mathbb{Z}_2$ (or $\mathbb{Z}$). Almost nothing is known in the general case.

...But Not the Algebra of Octonions

After the quaternions, the next "natural candidate" for commutativity is, of course, the algebra of octonions $\mathbb{O}$. However, let us show that:

The algebra $\mathbb{O}$ cannot be realized as a graded commutative algebra.

Indeed, recall that $\mathbb{O}$ contains 7 mutually anticommuting elements e_1, ...,e_7 such that $(e_\ell)^2 = -1$ for $\ell = 1, \ldots, 7$ that form several copies of $\mathbb{H}$ (see [2, 3] for a beautiful introduction to the octonions). Assume there is a grading $\sigma : e_\ell \mapsto \Gamma$ with values in an abelian group Γ, satisfying (2) and (3). Then, for three elements $e_{\ell_1}, e_{\ell_2}, e_{\ell_3} \in \mathbb{O}$, such that $e_{\ell_1} \cdot e_{\ell_2} = e_{\ell_3}$, one has

$$\sigma(e_{\ell_3}) = \sigma(e_{\ell_1}) + \sigma(e_{\ell_2}).$$

If now e_{ℓ_4} anticommutes with e_{ℓ_1} and e_{ℓ_2}, then e_{ℓ_4} has to commute with e_{ℓ_3} because of the linearity of the scalar product. This readily leads to a contradiction.

Multiplying the Triplets

Let us now take another look at the grading (1). It turns out that there is a simple way to reconstitute the whole structure of $\mathbb{H}$ directly from this formula.

First of all, we rewrite the grading as follows:

$$\begin{aligned} 1 &\leftrightarrow (0,\, 0,\, 0), \\ i &\leftrightarrow (0,\, 1_2, 1_3), \\ j &\leftrightarrow (1_1, 0,\, 1_3), \\ k &\leftrightarrow (1_1, 1_2, 0). \end{aligned} \tag{4}$$

Second of all, we define the rule for multiplication of triplets. This multiplication is nothing but the usual operation in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., the componentwise addition (modulo 2), for instance,

$$(1_1, 0, 0) \cdot (1_1, 0, 0) = (0, 0, 0),$$
$$(1_1, 0, 0) \cdot (0, 1_2, 0) = (1_1, 1_2, 0),$$

but with an important additional *sign rule*. Whenever we have to exchange "left-to-right" two units, 1_n and 1_m with $n > m$, we put the "$-$" sign, for instance

$$(0, 1_2, 0) \cdot (1_1, 0, 0) = -(1_1, 1_2, 0),$$

since we exchanged 1_2 and 1_1.

One then has for the triplets in (4):

$$i \cdot j \leftrightarrow (0, 1_2, 1_3) \cdot (1_1, 0, 1_3) = (1_1, 1_2, 0) \leftrightarrow k,$$

since the total number of exchanges is *even* (1_2 and 1_3 were exchanged with 1_1) and

$$j \cdot i \leftrightarrow (1_1, 0, 1_3) \cdot (0, 1_2, 1_3) = -(1_1, 1_2, 0) \leftrightarrow -k,$$

since the total number of exchanges is *odd* (1_3 was exchanged with 1_2). In this way, one immediately recovers the complete multiplication table of $\mathbb{H}$.

Remark 3.1 The above realization is, of course, related to the embedding of $\mathbb{H}$ into the associative algebra with 3 generators $\varepsilon_1, \varepsilon_2, \varepsilon_3$ subject to the relations

$$\varepsilon_n^2 = 1, \quad \varepsilon_n \varepsilon_m = -\varepsilon_m \varepsilon_n, \quad \text{for } n \neq m.$$

This embedding is given by

$$i \mapsto \varepsilon_2 \varepsilon_3, \quad j \mapsto \varepsilon_1 \varepsilon_3, \quad k \mapsto \varepsilon_1 \varepsilon_2$$

and is well known.

Everybody knows the famous story of Hamilton and his son asking his father the same question every morning: "Well, Papa, can you multiply triplets?" and always getting the same answer: "No, I can only add and subtract them", with a sad shake of the head. This story now has a happy ending. As we have just seen, Hamilton did nothing but multiply the triplets. Or should we rather say added and subtracted them?

ACKNOWLEDGMENTS

We are grateful to S. Tabachnikov for helpful suggestions.

REFERENCES

1. H. Albuquerque, S. Majid, *Quasialgebra structure of the octonions*, J. Algebra **220** (1999), 188–224.
2. J. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), 145–205.
3. J. Conway, D. Smith, On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry, A K Peters, Ltd., Natick, MA, 2003.
4. T. Y. Lam, *Hamilton's quaternions*, Handbook of Algebra, Vol. 3, 429–454, North-Holland, Amsterdam, 2003.

Université Paris Diderot Paris 7
UFR de mathématiques case 7012
75205 Paris Cedex 13
France
e-mail: sophiemg@umich.edu

CNRS, Institut Camille Jordan
Université Claude Bernard Lyon 1
43 boulevard du 11 novembre 1918,
69622 Villeurbanne Cedex
France
e-mail: ovsienko@math.univ-lyon1.fr